MATHEMATICS

Grades Pre-Kindergarten to 12

Massachusetts Curriculum Framework – 2017

MASSACHUSETTS DEPARTMENT OF
ELEMENTARY AND SECONDARY
EDUCATION

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Dear Colleagues,

I am pleased to present to you the Massachusetts Curriculum Framework for Mathematics adopted by the Board of Elementary and Secondary Education in March 2017. This Framework builds upon the foundation of the 2010 Massachusetts Curriculum Framework for Mathematics as well as versions of the Massachusetts Mathematics Framework published since 1995.

The current Framework incorporates improvements suggested by Massachusetts educators after six years of experience in implementing the 2010 standards in their classrooms. These revised pre-kindergarten to grade 12 standards are based on research and effective practice, and will enable teachers and administrators to strengthen curriculum, instruction, and assessment.

The 2017 standards draw from the best of prior Massachusetts standards, and represent the input of hundreds of the Commonwealth's K–12 and higher education faculty. The 2017 standards present the Commonwealth's commitment to providing all students with a world-class education.

This revision of the Framework retains the strengths of previous frameworks and includes these improved features:

- Increased coherence across the grades and improved clarity of mathematical terms and language to describe expectations for students.
- Clear expectations for student mastery of basic addition, subtraction, multiplication, and division facts.
- An enhanced high school section that includes: 1) clearer model course standards; and 2) guidance on making decisions for course sequences and the model Algebra I course, along with options for various course-taking pathways.
- Guidance for moving students into an Algebra I course by grade 8 and through Calculus in high school; and
- More detailed descriptions about rigor and aspirations for students with a stronger emphasis on how the content standards, the standards for mathematical practice, and the guiding principles prepare students for college, careers, and civic participation.

In the course of revising these standards, the Department received many valuable comments and suggestions. I want to thank everyone who contributed their suggestions and ideas to make these revised standards useful for educators, students, families, and the community. In particular, I am grateful to the members of the Mathematics Standards Review Panel and our Content Advisors for giving their time generously to the project to improve the learning standards for Massachusetts students. I am proud of the work that has been accomplished.

We will continue to collaborate with educators to implement the 2017 Massachusetts Curriculum Framework for Mathematics.

Thank you again for your ongoing support and for your commitment to achieving the goal of improved student achievement for all our students.

Sincerely,

Mitchell D. Chester, Ed. D.
Commissioner of Elementary and Secondary Education

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Introduction

The Origin of these Standards: 1993–2010

The Massachusetts Education Reform Act of 1993 directed the Commissioner and the Department of Elementary and Secondary Education[1] to create academic standards in a variety of subject areas. Massachusetts adopted its first set of Mathematics standards in 1995 and revised them in 2000. In 2007, the Massachusetts Department of Elementary and Secondary Education (ESE) convened a team of educators to revise its 2000 *Mathematics Curriculum Framework.* In 2009 the Council of Chief State School Officers (CCSSO) and the National Governors Association (NGA) began a multi-state standards development project called the *Common Core State Standards* initiative, whereupon the two efforts merged. The pre-kindergarten to grade 12 *Massachusetts Curriculum Framework for Mathematics*, a new framework that included both the Common Core State Standards and unique Massachusetts standards and features, was adopted by the Boards of Elementary and Secondary Education and Early Education and Care in 2010. A similar process unfolded for the English Language Arts/Literacy Framework.

Review of Mathematics and English Language Arts/Literacy Standards, 2016–2017

In November 2015, the Massachusetts Board of Elementary and Secondary Education voted to move forward with the development of its own next generation student assessment program in mathematics and English Language Arts/Literacy. At the same time, the Board supported a plan to convene review panels comprised of Massachusetts K-12 educators and higher education faculty to review the 2010 Mathematics and English Language Arts/Literacy Curriculum Frameworks. The review panels were also asked to identify any modifications or additions to ensure that the Commonwealth's standards match those of the most aspirational education systems in the world, thus representing a course of study that best prepares students for the 21st century.

In February 2016, the Department appointed a panel of Massachusetts educators from elementary, secondary, and higher education to review the mathematics and ELA/Literacy standards and suggest improvements based on their experiences using the 2010 Framework. The Department sought comment on the standards through a public survey and from additional content advisors in mathematics and ELA/Literacy.

The 2017 *Massachusetts Curriculum Framework for Mathematics* reflects improvements to the prior Framework that have been informed by the review panel, public comments, and the content advisors. In some cases, the standards have been edited to clarify meaning. Some have been eliminated; others added. The Glossary and Bibliography have been updated and the Department's 2010 document titled, *Making Decisions about Course Sequences and the Model Algebra I Course*, has been revised and is now included in the high school section of the Framework. The intent is to present multiple pathways involving the compression or enhancement of mathematics standards to provide alternative course-taking sequences for students enabling them to be successful and prepared for various college and career pursuits, including mathematic-intensive majors and careers.

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[1] At the time, the agency was called the Department of Education.

The 2017 standards draw from the best of prior Massachusetts standards, and represent the input of hundreds of the Commonwealth's K-12 and higher education faculty. The 2017 standards present the Commonwealth's commitment to providing all students with a world-class education.

The Mathematically Proficient Person of the Twenty-First Century

The standards describe a vision of what it means to be a mathematically proficient person in this century. Students who are college and career ready in mathematics will at a minimum demonstrate the academic knowledge, skills, and practices necessary to enter into and succeed in entry-level, credit bearing courses in College Algebra, Introductory College Statistics, or technical courses. It also extends to a comparable entry-level course or a certificate or workplace training program requiring an equivalent level of mathematics. At the same time, the standards provide for a course of study that will prepare students for a science, technology, engineering, or mathematics career. For example, the level of mathematics preparation necessary to succeed in an engineering program is more ambitious than the preparation needed to succeed in an entry-level, credit-bearing mathematics course as part of a liberal arts program. The standards provide pathways for students who want to pursue a mathematics-intensive career or academic major after high school.

The mathematical skills and understanding that students are expected to demonstrate have wide applicability outside the classroom or workplace. Students who meet the standards are able to identify problems, represent problems, justify conclusions, and apply mathematics to practical situations. They gain understanding of topics and issues by reviewing data and statistical information. They develop reasoning and analytical skills and make conclusions based on evidence that is essential to both private deliberation and responsible citizenship in a democratic society. They understand mathematics as a language for representing the physical world.

They are able to use and apply their mathematical thinking in various contexts and across subject areas, for example, in managing personal finances, designing a robot, or presenting a logical argument and supporting it with relevant quantitative data in a debate. Students should be given opportunities to discuss math's relevance to everyday life and their interests and potential careers with teachers, parents, business owners, and employees in a variety of fields such as computer science, architecture, construction, healthcare, engineering, retail sales, and education. From such discussions, students can learn that a computer animator uses linear algebra to determine how an object will be rotated, shifted, or altered in size. They can discover that an architect uses math to calculate the square footage of rooms and buildings, to lay out floor dimensions and to calculate the required space for areas such as parking or heating and cooling systems (kumon.org}. They can investigate how public policy analysts use statistics to monitor and predict state, national or international healthcare use, benefits, and costs.

Students who meet the standards develop persistence, conceptual understanding, and procedural fluency; they develop the ability to reason, prove, justify, and communicate. They build a strong foundation for applying these understandings and skills to solve real world problems. These standards represent an ambitious pre-kindergarten to grade 12 mathematics program that ensure that students are prepared for college, careers, and civic life.

A Coherent Progression of Learning: Pre-Kindergarten through Grade 12

The mathematics content standards are presented by individual grade levels in pre-kindergarten through grade 8 to provide useful specificity. The pre-kindergarten standards apply to children who are four-year-olds and younger five-year-olds. A majority of these students attend education programs in a variety of settings:

community-based early care and education centers; family daycare; Head Start programs; and public preschools. In this age group, the foundations of counting, quantity, comparing shapes, adding, and taking apart – and the ideas that objects can be measured – are formed during conversations, play, and with experiences with real objects and situations.

At the high school level, the standards are presented in two different ways:

1. **Conceptual Categories:** These categories portray a coherent view of high school mathematics standards for the grade-span 9–12. These content standards are organized into six categories: Number and Quantity; Algebra; Functions; Modeling; Geometry; and Statistics and Probability.

2. **Model High School Courses:** Two high school pathways, the Traditional Pathway Model Courses (Algebra I, Geometry, Algebra II) and the Integrated Pathway Model Courses (Mathematics I, II, III), are presented using the content standards in the Conceptual Categories. These model courses were designed to create a smooth transition from the grade-by-grade pre-k–8 content standards to high school courses. All of the college and career ready content standards are included in appropriate locations within each set of three model high school courses. The high school content standards coded with a (+) symbol are optional and identify higher-level mathematics skills and knowledge that students should learn in order to take more advanced mathematics courses. In addition, two Advanced Model High School Courses, Advanced Quantitative Reasoning and Precalculus, are included. Students may choose to take these courses after completing either of the Model High School Pathways. See the section below titled, "Course-Taking Sequences and Pathways for All Students," for additional advanced mathematics courses and pathways students might pursue in high school.

Focus, Clarity, and Rigor

In the past, mathematics standards and curricula were often criticized for being "a mile wide and an inch deep" in almost every topic taught each year. The 2010 Framework presented a new design feature for grades Pre-K–12: a focus on three to five critical focus areas per grade or course. The 2017 Framework continues to concentrate on fewer topics in each grade to allow students to deepen and consolidate their understanding in these areas. These critical focus areas are useful in communicating with families and the community and for designing curricula, support services, and programs.

A Balance of Conceptual Understanding, Procedural Fluency, and Application[2]

The standards strategically develop students' mathematical understanding and skills. When students are first introduced to a mathematical concept they explore and investigate the concept by using concrete objects, visual models, drawings, or representations to build their understanding. In the early grades they develop number sense while working with numbers in many ways. They learn a variety of strategies to solve problems and use what they have learned about patterns in numbers and the properties of numbers. This serves to develop a strong understanding of number sense, decomposing and composing numbers, the relationship between addition and subtraction, and multiplication and division. In calculations, students are expected to be able to use the most efficient and accurate way to solve a problem based on their understanding and knowledge of place value and properties of numbers. Students reach fluency by building understanding of mathematical concepts –

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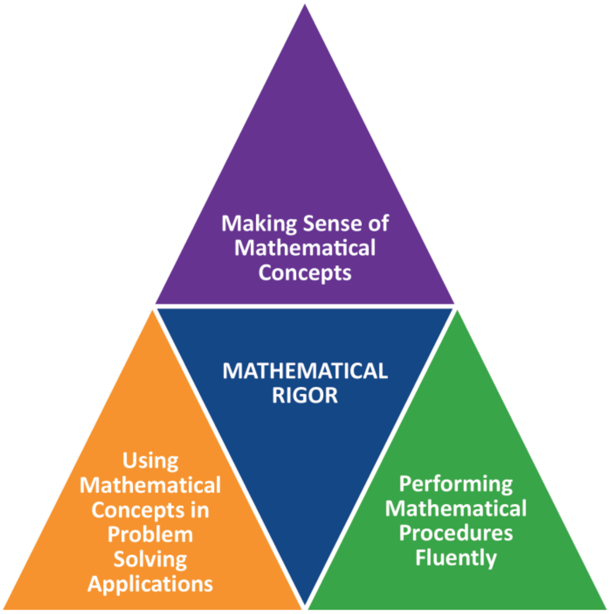
[2] Sealey, Cathy. *Balance is Basic, A 21st Century View of a Balanced Mathematical Program*

this lays a strong foundation that prepares students for more advanced math work – and by building automaticity in the recall of basic computation facts, such as addition, subtraction, multiplication, and division.

As students apply their mathematical knowledge and skills to solve real-world problems, they also gain an understanding of the importance of mathematics throughout their lives.

To achieve mathematical understanding, students should be actively engaged in meaningful mathematics. The content and practice standards focus on developing students in the following areas:



- **Conceptual understanding** – make sense of the math, reason about and understand math concepts and ideas
- **Procedural fluency** – know mathematical facts, compute and do the math
- **Capacity** – solve a wide range of problems in various contexts by reasoning, thinking, and **applying** the mathematics they have learned.

Middle and High School Course-Taking Sequences and Pathways for All Students

The Massachusetts High School Program of Studies (MassCore) is a recommended program of studies that includes four years of mathematics coursework in grades 9–12. MassCore describes other learning opportunities, such as Advanced Placement (AP) classes, dual enrollment, senior projects, online courses for high school or college credit, and service or work-based learning.

The Framework provides an opportunity for districts to revisit and plan course sequences in middle and high school mathematics along with educators, middle and high school guidance counselors, parents, college mathematics faculty, and mathematics leaders. This Framework includes a new section titled, *Making Decisions about Course Sequences and the Model Algebra I Course*. This section presents several options for pathways for students ready to move at an accelerated rate.

On Grade Sequence: Students who follow the Framework grade-level sequence pre-K–8 will be prepared for the Traditional or Integrated Model Course high school pathway, beginning with Algebra I or Mathematics I in grade 9. Students following this pathway will be prepared to take a fourth year advanced course in grade 12, such as the Model Precalculus Course, Model Quantitative Reasoning Course, or other advanced courses offered in their district.

Model Algebra I in Grade 8 and High School Acceleration: One option for accelerating learning is to take the Model Algebra I course in grade 8. This pathway option compresses the standards for grade 7 with part of the grade 8 standards in grade 7. That allows grade 8 students to complete the grade 8 standards related to algebra and the Algebra I model high school standards in one year (grade 8). Considerations for assigning a student this pathway include two factors: grade 8 standards are already rigorous and students are expected to learn the grade 8 standards in order to be prepared for the Algebra I model course.

This section also presents pathways for students who are ready to accelerate their learning, beginning in grade 9. Some of these pathways lead to calculus in grade 12 while others offer a sequence to other advanced courses such as Quantitative Reasoning, Statistics, Linear Algebra, AP courses, Discrete Mathematics, or participating in a dual enrollment program.

All pathways should aspire to meet the goal of ensuring that no student who graduates from a Massachusetts High School will be placed into a remedial mathematics course in a Massachusetts public college or university. Achieving this goal may require mathematics secondary educators and college faculty to work collaboratively to select or co-develop appropriate 12th grade coursework and assessments. Presenting a variety of course-taking pathways encourages students to persist in their mathematical studies. It also helps them realize that there are multiple opportunities to make course-taking decisions as they continue to advance mathematically and pursue their interests and career and college goals.

Mathematics in the Context of a Well-Rounded Curriculum

Strong mathematics achievement is a requisite for studying the sciences (including social sciences), technology (including computer science), and engineering. The centrality of mathematics to the pursuit of STEM careers is well documented.

In addition, an effective mathematics program builds upon and develops students' mathematical knowledge and literacy skills. Reading, writing, speaking, and listening skills are necessary elements of learning and engaging in mathematics, as well as in other content areas. The English Language Arts/Literacy standards require that instruction in reading, writing, speaking, listening, and language is a shared responsibility within the school. The pre-K–5 ELA/Literacy standards include expectations for reading, writing, speaking, listening, and language applicable to a range of subjects, including mathematics, social studies, science, the arts, and comprehensive health. Grades 6–12 ELA/Literacy standards are divided into two sections, one for ELA and the other for history/social studies, science, mathematics, and technical subjects. This division reflects the unique, time-honored role of ELA teachers in developing students' literacy skills, while at the same time recognizing that teachers in other disciplines also contribute in this development.

Consistent with emphasizing the importance of an interdisciplinary approach to literacy, the Mathematics Guiding Principles recognize that reading, writing, speaking, and listening skills are necessary elements of learning and engaging in mathematics. Mathematics students learn specialized vocabulary, terms, notations, symbols, representations, and models relevant to the grade level. Being able to read, interpret, and analyze mathematical information from a variety of sources and communicating mathematically in written and oral forms are critical skills to college and career readiness, citizenship, and informed decision-making.

In essence, mathematics is a language for describing and understanding the physical world. Notably, the recent revision of the *Massachusetts Curriculum Framework for Science and Technology/Engineering* (2016) also highlights literacy in its Guiding Principles and Practices.

To achieve a well-rounded curriculum at all grade levels, the standards in this Framework are meant to be used with the *Massachusetts Curriculum Framework for English Language Arts/Literacy, the Arts, History and Social Science, Science and Technology/Engineering, Comprehensive Health and Physical Education, Foreign Languages*.

In grades 9–12, the standards are also meant to be used with the *Framework for Career and Vocational and Technical Education* to achieve a truly rich and well-rounded curriculum.

What the Mathematics Curriculum Framework Does and Does Not Do

The standards define what all students are expected to know and be able to do, not how teachers should teach. While the standards focus on what is most essential, they do not describe all that can or should be taught. A great deal is left to the discretion of teachers and curriculum developers.

No set of grade-level standards can reflect the great variety of abilities, needs, learning rates, and achievement levels in any given classroom. The standards define neither the support materials that some students may need, nor the advanced materials that others should have access to. It is also beyond the scope of the standards to define the full range of support appropriate for English learners and for students with disabilities. Still, all students must have the opportunity to learn and meet the same high standards if they are to access the knowledge and skills that will be necessary in their post-high-school lives.

The standards should be read as allowing for the widest possible range of students to participate fully from the outset with appropriate accommodations to ensure maximum participation of students with special education needs. For example, for students with disabilities, *reading* math texts and problems should allow for the use of Braille, screen-reader technology, or other assistive devices. *Writing* should include the use of a scribe, computer, or speech-to-text technology that includes mathematical terms, notations, and symbols. In a similar manner, *speaking* and *listening* should be interpreted broadly to include sign language (see Appendix 1).

While the mathematics described herein is critical to college, career, and civic readiness, they do not define the "whole" of readiness. Students require a wide-ranging, rigorous academic preparation and attention to such matters as social, emotional, and physical development and approaches to learning.

Document Organization

Eight Guiding Principles for Mathematical Programs in Massachusetts follows this introductory section. The Guiding Principles are philosophical statements that underlie the standards and resources in this Framework.

Following the Guiding Principles are the eight **Standards for Mathematical Practice.** These standards describe the varieties of expertise that all mathematics educators at all levels should seek to develop in their students.

Following the Standards for Mathematical Practice are the **Standards for Mathematical Content,** presented in three sections:

1. **Pre-kindergarten through grade 8 content standards by *grade level***
2. **High school content standards by *conceptual category***
3. **High school content standards by *model high school courses***—includes six model courses outlined in two pathways (Traditional and Integrated) and two model advanced courses, Precalculus and Advanced Quantitative Reasoning.

As described above, this Framework also includes a section in the high school content standards, entitled "Making Course Decisions about Course Sequences and the Model High School Algebra I Course." This new section provides options for middle and high school course-taking sequences, including pathways that accelerate learning in order to allow students to reach advanced courses, such as Calculus, by the end of grade 12.

The supplementary resources that follow the learning standards address both engaging learners in content through the Standards for Mathematical Practice, and guidance in applying the standards for English language learners and students with disabilities. A Glossary of mathematical terms, tables, illustrations, and a list of references is also included.

Guiding Principles for Mathematics Programs in Massachusetts

The following principles are philosophical statements that underlie the pre-kindergarten through grade 12 mathematics standards and resources presented in this Framework. These principles should guide the design and evaluation of mathematics programs. Programs guided by these principles will prepare students for colleges, careers, and their lives as productive citizens.

Guiding Principle 1

Educators must have a deep understanding of the mathematics they teach, not only to help students learn how to efficiently do mathematical calculations, but also to help them understand the fundamental principles of mathematics that are the basis for those operations. Teachers should work with their students to master these underlying concepts and the relationships between them in order to lay a foundation for higher-level mathematics, strengthen their capacity for thinking logically and rigorously, and develop an appreciation for the beauty of math.

Guiding Principle 2

To help all students develop a full understanding of mathematical concepts and procedural mastery, educators should provide them with opportunities to apply their learning and solve problems using multiple methods, in collaboration with their peers and independently, and complemented by clear explanations of the underlying mathematics.

Guiding Principle 3

Students should have frequent opportunities to discuss and write about various approaches to solving problems, in order to help them develop and demonstrate their mathematical knowledge, while drawing connections between alternative strategies and evaluating their comparative strengths and weaknesses.

Guiding Principle 4

Students should be asked to solve a diverse set of real-world and other mathematical problems, including equations that develop and challenge their computational skills, and word problems that require students to design their own equations and mathematical models. Students learn that with persistence, they can solve challenging problems and be successful.

Guiding Principle 5

A central part of an effective mathematics curriculum should be the development of a specialized mathematical vocabulary including notations and symbols, and an ability to read and understand mathematical texts and information from a variety of sources.

Guiding Principle 6

Assessment of student learning should be a daily part of a mathematics curriculum to ensure that students are progressing in their knowledge and skill, and to provide teachers with the information they need to adjust their instruction and differentiate their support of individual students.

Guiding Principle 7

Students at all levels should have an opportunity to use appropriate technological tools to communicate ideas, provide a dynamic approach to mathematic concepts, and to search for information. Technological tools can also be used to improve efficiency of calculation and enable more sophisticated analysis, without sacrificing operational fluency and automaticity.

Guiding Principle 8

Social and emotional learning can increase academic achievement, improve attitudes and behaviors, and reduce emotional distress. Students should practice self-awareness, self-management, social awareness, responsible decision-making, and relationship skills, by, for example: collaborating and learning from others and showing respect for others' ideas; applying the mathematics they know to make responsible decisions to solve problems, engaging and persisting in solving challenging problems; and learning that with effort, they can continue to improve and be successful.

The Standards for Mathematical Practice

The Standards for Mathematical Practice describe expertise that mathematics educators at all levels should seek to develop in their students. The Standards for Mathematical Practice describe ways in which developing student practitioners of the discipline of mathematics engage with the subject matter as they grow in mathematical maturity and expertise throughout the elementary, middle, and high school years. These practices rest on two sets of important "processes and proficiencies" that have longstanding importance in mathematics education—the National Council of Teachers of Mathematics (NCTM) process standards and the strands of mathematical proficiency specified in the National Research Council's Report "Adding It Up."

Designers of curricula, assessments, and professional development should endeavor to connect the mathematical practices to mathematical content in instruction.

1. Make sense of problems and persevere in solving them.

Mathematically proficient students start by explaining to themselves the meaning of a problem and looking for entry points to its solution. They analyze givens, constraints, relationships, and goals. They make conjectures about the form and meaning of the solution and plan a solution pathway rather than simply jumping into a solution attempt. They consider analogous problems, and try special cases and simpler forms of the original problem in order to gain insight into its solution. They monitor and evaluate their progress and change course if necessary. Older students might, depending on the context of the problem, transform algebraic expressions or change the viewing window on their graphing calculator to get the information they need. Mathematically proficient students can explain correspondences between equations, verbal descriptions, tables, and graphs or draw diagrams of important features and relationships, graph data, and search for regularity or trends. Younger students might rely on using concrete objects or pictures to help conceptualize and solve a problem. Mathematically proficient students check their answers to problems using a different method, and they continually ask themselves, "Does this make sense?" They can understand others' approaches to solving complex problems and identify correspondences among different approaches.

2. Reason abstractly and quantitatively.

Mathematically proficient students make sense of the quantities and their relationships in problem situations. Students bring two complementary abilities to bear on problems involving quantitative relationships: the ability to decontextualize—to abstract a given situation and represent it symbolically, and manipulate the representing symbols as if they have a life of their own, without necessarily attending to their referents—and the ability to contextualize, to pause as needed during the manipulation process in order to probe into the referents for the symbols involved. Quantitative reasoning entails habits of creating a coherent representation of the problem at hand; considering the units involved; attending to the meanings of quantities, not just how to compute them; and knowing and flexibly using different properties of operations and objects.

3. Construct viable arguments and critique the reasoning of others.

Mathematically proficient students understand and use stated assumptions, definitions, and previously established results in constructing arguments. They make conjectures and build a logical progression of statements to explore the truth of their conjectures. They are able to analyze situations by breaking them into cases, and can recognize and use counterexamples. They justify their conclusions, communicate them to others, and respond to the arguments of others. They reason inductively about data, making plausible arguments that take into account the context from which the data arose. Mathematically proficient students are also able to compare the effectiveness of two plausible arguments, distinguish correct logic or reasoning from that which is flawed, and—if there is a flaw in an argument—explain what it is. Elementary students can construct arguments using concrete referents such as objects, drawings, diagrams, and actions. Such arguments can make sense and be correct, even though they are not generalized or made formal until later grades. Later, students learn to determine domains to which an argument applies. Students at all grades can listen or read the arguments of others, decide whether they make sense, and ask useful questions to clarify or improve the arguments.

4. Model with mathematics.

Mathematically proficient students can apply the mathematics they know to solve problems arising in everyday life, society, and the workplace. In early grades, this might be as simple as writing an addition equation to describe a situation. In middle grades, a student might apply proportional reasoning to plan a school event or analyze a problem in the community. By high school, a student might use geometry to solve a design problem or use a function to describe how one quantity of interest depends on another. Mathematically proficient students who can apply what they know are comfortable making assumptions and approximations to simplify a complicated situation, realizing that these may need revision later. They are able to identify important quantities in a practical situation and map their relationships using such tools as diagrams, two-way tables, graphs, flowcharts and formulas. They can analyze those relationships mathematically to draw conclusions. They routinely interpret their mathematical results in the context of the situation and reflect on whether the results make sense, possibly improving the model if it has not served its purpose.

5. Use appropriate tools strategically.

Mathematically proficient students consider the available tools when solving a mathematical problem. These tools might include pencil and paper, concrete models, a ruler, a protractor, a calculator, a spreadsheet, a computer algebra system, a statistical package, or dynamic geometry software. Proficient students are sufficiently familiar with tools appropriate for their grade or course to make sound decisions about when each of these tools might be helpful, recognizing both the insight to be gained and their limitations. For example, mathematically proficient high school students analyze graphs of functions and solutions generated using a graphing calculator. They detect possible errors by strategically using estimation and other mathematical knowledge. When making mathematical models, they know that technology can enable them to visualize the results of varying assumptions, explore consequences, and compare predictions with data. Mathematically proficient students at various grade levels are able to identify relevant external mathematical resources, such as digital content located on a website, and use them to pose or solve problems. They are able to use technological tools to explore and deepen their understanding of concepts.

6. Attend to precision.

Mathematically proficient students try to communicate precisely to others. They try to use clear definitions in discussion with others and in communicating their own reasoning verbally and/or in writing. In problem solving they state the meaning of the symbols they choose, including using the equal sign consistently and appropriately. They are careful about specifying units of measure and labeling axes to clarify the correspondence with quantities in a problem. They calculate accurately and efficiently, expressing numerical answers with a degree of precision appropriate for the problem context. In the elementary grades, students give carefully formulated explanations to each other. By the time they reach high school, they have learned to examine claims and make explicit use of definitions.

7. Look for and make use of structure.

Mathematically proficient students look closely to discern a pattern or structure. Young students, for example, might notice that three and seven more is the same amount as seven and three more, or they may sort a collection of shapes according to how many sides the shapes have. Later, students will see 7 × 8 equals the well-remembered 7 × 5 + 7 × 3, in preparation for learning about the distributive property. In the expression $x^2 + 9x + 14$, older students can see the 14 as 2 × 7 and the 9 as 2 + 7. They recognize the significance of an existing line in a geometric figure and can use the strategy of drawing an auxiliary line for solving problems. They also can step back for an overview and shift perspective. They can see complicated things, such as some algebraic expressions, as single objects or as being composed of several objects. For example, they can see $5 - 3(x - y)^2$ as 5 minus a positive number times a square, and use that to realize that its value cannot be more than 5 for any real numbers x and y.

8. Look for and express regularity in repeated reasoning.

Mathematically proficient students notice if calculations are repeated, and look both for general methods and for shortcuts. Upper elementary students might notice when dividing 25 by 11 that they are repeating the same calculations over and over again, and conclude they have a repeating decimal. By paying attention to the calculation of slope as they repeatedly check whether points are on the line through (1, 2) with slope 3, middle school students might abstract the equation $^{(y-2)}/_{(x-1)} = 3$. Noticing the regularity in the way terms cancel when expanding $(x - 1)(x + 1)$, $(x - 1)(x^2 + x + 1)$, and $(x - 1)(x^3 + x^2 + x + 1)$ might lead them to the general formula for the sum of a geometric series. As they work to solve a problem, mathematically proficient students maintain oversight of the process, while attending to the details. They continually evaluate the reasonableness of their intermediate results.

Standards for Mathematical Content Pre-Kindergarten–Grade 8

ORGANIZATION OF THE K–8 STANDARDS

Pre-Kindergarten

Kindergarten

Grade 1

Grade 2

Grade 3

Grade 4

Grade 5

Grade 6

Grade 7

Grade 8

Organization of the Pre-Kindergarten to Grade 8 Content Standards

The pre-kindergarten through grade 8 content standards in this Framework are organized by **grade level**. Within each grade level, standards are grouped first by **domain**. Each domain is further subdivided into **clusters** of related standards.

- **Standards** define what students should understand and be able to do.
- **Clusters** are groups of related standards. Note that standards from different clusters may sometimes be closely related, because mathematics is a connected subject.
- **Domains** are larger groups of related standards. Standards from different domains may sometimes be closely related.

The table below shows which domains are addressed at each grade level:

Progression of Pre-K–8 Domains										
Domain	**Grade Level**									
	PK	**K**	**1**	**2**	**3**	**4**	**5**	**6**	**7**	**8**
Counting and Cardinality	■	■								
Operations and Algebraic Thinking	■	■	■	■	■	■	■			
Number and Operations in Base Ten		■	■	■	■	■	■			
Number and Operations – Fractions					■	■	■			
The Number System								■	■	■
Ratios and Proportional Relationships								■	■	
Expressions and Equations								■	■	■
Functions										■
Measurement and Data	■	■	■	■	■	■	■			
Geometry	■	■	■	■	■	■	■	■	■	■
Statistics and Probability								■	■	■

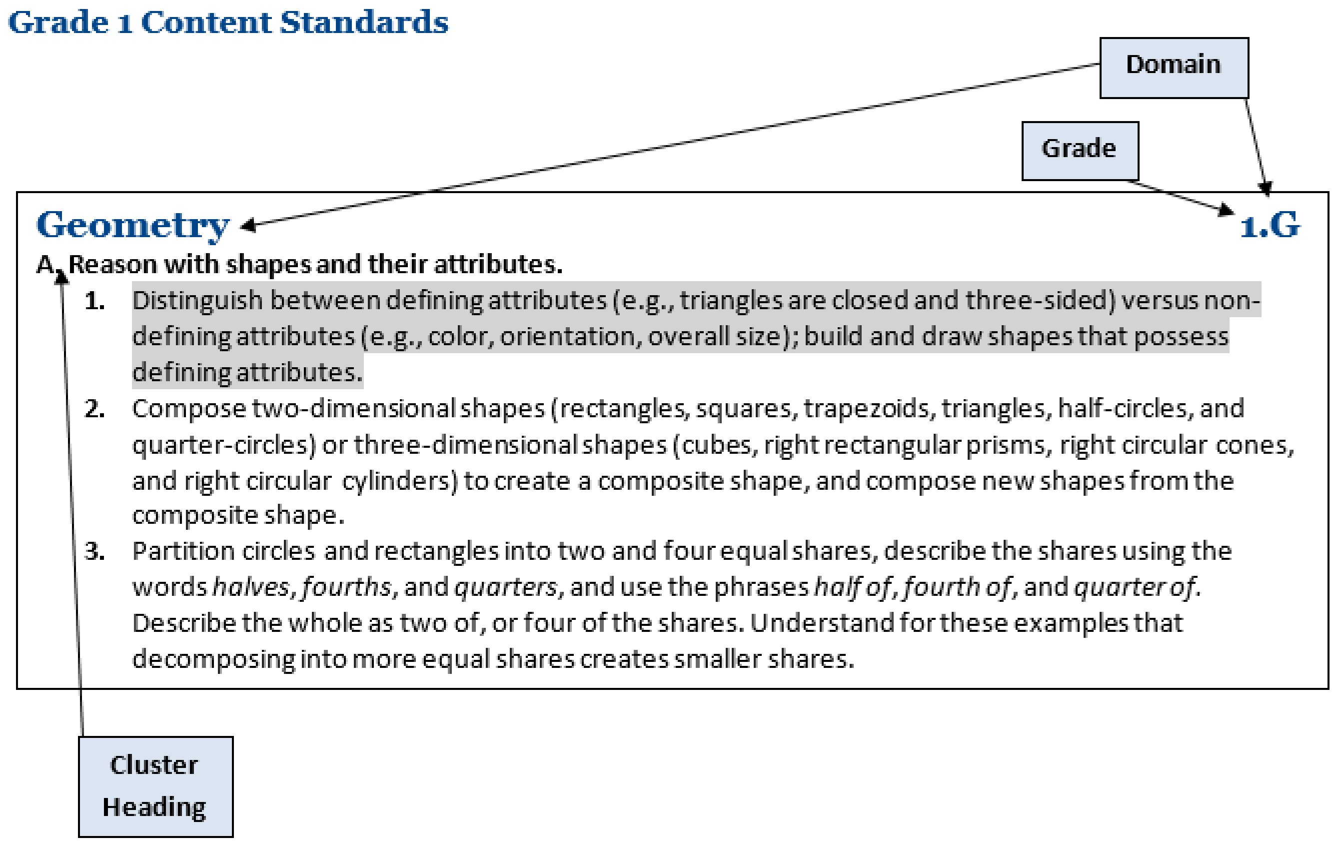
Format for Each Grade Level

Each grade level is presented in the same format:

- An introduction and description of the critical areas for learning at that grade.
- An overview of that grade's domains and clusters.
- The content standards for that grade (presented by domain, cluster heading, and individual standard).

Standards Identifiers/Coding

Each standard has a unique identifier that consists of the grade level, (PK, K, 1, 2, 3, 4, 5, 6, 7, or 8), the domain code, and the standard number, as shown in the example below.



Grade 1 Content Standards

Geometry **1.G**

A. Reason with shapes and their attributes.

1. Distinguish between defining attributes (e.g., triangles are closed and three-sided) versus non-defining attributes (e.g., color, orientation, overall size); build and draw shapes that possess defining attributes.
2. Compose two-dimensional shapes (rectangles, squares, trapezoids, triangles, half-circles, and quarter-circles) or three-dimensional shapes (cubes, right rectangular prisms, right circular cones, and right circular cylinders) to create a composite shape, and compose new shapes from the composite shape.
3. Partition circles and rectangles into two and four equal shares, describe the shares using the words *halves*, *fourths*, and *quarters*, and use the phrases *half of*, *fourth of*, and *quarter of*. Describe the whole as two of, or four of the shares. Understand for these examples that decomposing into more equal shares creates smaller shares.

The first standard above is identified as 1.G.A.1, identifying it as a grade 1 standard in the Geometry Domain, and as the first standard in that domain. Standard 1.G.A.1 is the first standard in this cluster of standards. All of the standards in this Framework use a common coding system.

Pre-Kindergarten

Introduction

The pre-kindergarten standards presented by Massachusetts are guideposts to facilitate young children's underlying mathematical understanding. The Massachusetts pre-kindergarten standards apply to children who are in the age group of older four- and younger five-year olds. The standards—which correspond with the learning activities in the *Massachusetts Guidelines for Preschool Learning Experiences* (2003)—can be promoted through play and exploration activities, and embedded in almost all daily activities. They should not be limited to "math time." In this age group, foundations of mathematical understanding are formed out of children's experiences with real objects and materials.

In preschool or pre-kindergarten, activity time should focus on two critical areas: (1) developing an understanding of whole numbers to 10, including concepts of one-to-one correspondence, counting, cardinality (the number of items in a set), and comparison; and (2) recognizing two-dimensional shapes, describing spatial relationships, and sorting and classifying objects by one or more attributes. Relatively more learning time should be devoted to developing children's sense of number as quantity than to other mathematics topics.

1. Young children begin counting and quantifying numbers up to 10. They begin with oral counting and recognition of numerals and word names for numbers. Experience with counting naturally leads to quantification. Children count objects and learn that the sizes, shapes, positions, or purposes of objects do not affect the total number of objects in the group. One-to-one correspondence matches each element of one set to an element of another set, providing a foundation for the comparison of groups and the development of comparative language such as *more than, less than,* and *equal to*.
2. Young children explore shapes and the relationships among them. They identify the attributes of different shapes, including length, area, and weight, by using vocabulary such as *long, short, tall, heavy, light, big, small, wide, narrow*. They compare objects using comparative language such as *longer/shorter, same length, heavier/lighter*. They explore and create two- and three-dimensional shapes by using various manipulative and play materials such as Popsicle sticks, blocks, pipe cleaners, and pattern blocks. They sort, categorize, and classify objects and identify basic two-dimensional shapes using the appropriate language.

The Standards for Mathematical Practice complement the content standards so that students increasingly engage with the subject matter as they grow in mathematical maturity and expertise throughout the elementary, middle, and high school years.

Pre-Kindergarten Overview

Counting and Cardinality

A. Know number names and the counting sequence.
B. Count to tell the number of objects.
C. Compare numbers.

Operations and Algebraic Thinking

A. Understand addition as putting together and adding to, and understand subtraction as taking apart and taking from.

Measurement and Data

A. Describe and compare measurable attributes.
B. Classify objects and count the number of objects in each category.
C. Work with money.

Geometry

A. Identify and describe shapes (squares, circles, triangles, rectangles).
B. Analyze, compare, create, and compose shapes.

Standards for Mathematical Practice

1. Make sense of problems and persevere in solving them.
2. Reason abstractly and quantitatively.
3. Construct viable arguments and critique the reasoning of others.
4. Model with mathematics.
5. Use appropriate tools strategically.
6. Attend to precision.
7. Look for and make use of structure.
8. Look for and express regularity in repeated reasoning.

Pre-Kindergarten Content Standards

Counting and Cardinality PK.CC

A. Know number names and the counting sequence.

1. Listen to and say the names of numbers in meaningful contexts.
2. Recognize and name written numerals 0–10.

B. Count to tell the number of objects.

3. Understand the relationships between numerals and quantities up to 10.

C. Compare numbers.

4. Count many kinds of concrete objects and actions up to ten, using one-to-one correspondence, and accurately count as many as seven things in a scattered configuration. Recognize the "one more," "one less" patterns.
5. Use comparative language, such as *more/less than, equal to,* to compare and describe collections of objects.

Operations and Algebraic Thinking PK.OA

A. Understand addition as putting together and adding to, and understand subtraction as taking apart and taking from.

1. Use concrete objects to model real-world addition (putting together) and subtraction (taking away) problems up through five.

Measurement and Data PK.MD

A. Describe and compare measurable attributes.

1. Recognize the attributes of length, area, weight, and capacity of everyday objects using appropriate vocabulary (e.g., *long, short, tall, heavy, light, big, small, wide, narrow*).
2. Compare the attributes of length and weight for two objects, including longer/shorter, same length; heavier/lighter, same weight; holds more/less, holds the same amount.

B. Classify objects and count the number of objects in each category.

3. Sort, categorize, and classify objects by more than one attribute.

C. Work with money.

4. Recognize that certain objects are coins and that dollars and coins represent money.

Geometry PK.G

A. Identify and describe shapes (squares, circles, triangles, rectangles).

1. Identify relative positions of objects in space, and use appropriate language (e.g., *beside, inside, next to, close to, above, below, apart*).
2. Identify various two-dimensional shapes using appropriate language.

B. Analyze, compare, create, and compose shapes.

3. Create and represent three-dimensional shapes (ball/sphere, square box/cube, tube/cylinder) using various manipulative materials (such as Popsicle sticks, blocks, pipe cleaners, pattern blocks).

Kindergarten

Introduction

In kindergarten, instructional time should focus on two critical areas: (1) representing, relating, and operating on whole numbers, initially with sets of objects; and (2) describing shapes and space. More learning time in kindergarten should be devoted to number than to other topics.

1. Students use numbers, including written numerals, to represent quantities and to solve quantitative problems, such as counting objects in a set; counting out a given number of objects; comparing sets or numerals; and modeling simple joining and separating situations with sets of objects, or eventually with equations such as 5 + 2 = 7 and 7 – 2 = 5. (Kindergarten students should see addition and subtraction equations, and student writing of equations in kindergarten is encouraged, but it is not required.) Students choose, combine, and apply effective strategies for answering quantitative questions, including quickly recognizing the cardinalities of small sets of objects, counting and producing sets of given sizes, counting the number of objects in combined sets, or counting the number of objects that remain in a set after some are taken away.
2. Students describe their physical world using geometric ideas (e.g., shape, orientation, spatial relations) and vocabulary. They identify, name, and describe basic two-dimensional shapes, such as squares, triangles, circles, rectangles, and hexagons, presented in a variety of ways (e.g., with different sizes and orientations), as well as three-dimensional shapes such as cubes, cones, cylinders, and spheres. They use basic shapes and spatial reasoning to model objects in their environment and to construct more complex shapes.

The Standards for Mathematical Practice complement the content standards so that students increasingly engage with the subject matter as they grow in mathematical maturity and expertise throughout the elementary, middle, and high school years.

Kindergarten Overview

Counting and Cardinality

- **A.** Know number names and the counting sequence.
- **B.** Count to tell the number of objects.
- **C.** Compare numbers.

Operations and Algebraic Thinking

- **A.** Understand addition as putting together and adding to, and understand subtraction as taking apart and taking from.

Number and Operations in Base Ten

- **A.** Work with numbers 11–19 to gain foundations for place value.

Measurement and Data

- **A.** Describe and compare measurable attributes.
- **B.** Classify objects and count the number of objects in each category.

Geometry

- **A.** Identify and describe shapes (squares, circles, triangles, rectangles, hexagons, cubes, cones, cylinders, and spheres).
- **B.** Analyze, compare, create, and compose shapes.

Standards for Mathematical Practice

1. Make sense of problems and persevere in solving them.
2. Reason abstractly and quantitatively.
3. Construct viable arguments and critique the reasoning of others.
4. Model with mathematics.
5. Use appropriate tools strategically.
6. Attend to precision.
7. Look for and make use of structure.
8. Look for and express regularity in repeated reasoning.

Kindergarten Content Standards

Counting and Cardinality K.CC

A. Know number names and the count sequence.

1. Count to 100 by ones and by tens.
2. Count forward beginning from a given number within the known sequence (instead of having to begin at one).
3. Write numbers from 0 to 20. Represent a number of objects with a written numeral 0–20 (with 0 representing a count of no objects).

B. Count to tell the number of objects.

4. Understand the relationship between numbers and quantities; connect counting to cardinality.
    a. When counting objects, say the number names in the standard order, pairing each object with one and only one number name and each number name with one and only one object.
    b. Understand that the last number name said tells the number of objects counted. The number of objects is the same regardless of their arrangement or the order in which they were counted.
    c. Understand that each successive number name refers to a quantity that is one larger. Recognize the one more pattern of counting using objects.
5. Count to answer "how many?" questions about as many as 20 things arranged in a line, a rectangular array, or a circle, or as many as 10 things in a scattered configuration; given a number from 1–20, count out that many objects.

C. Compare numbers.

6. Identify whether the number of objects in one group is greater than, less than, or equal to the number of objects in another group for groups with up to 10 objects, e.g., by using matching and counting strategies.
7. Compare two numbers between 1 and 10 presented as written numerals.

Operations and Algebraic Thinking K.OA

A. Understand addition as putting together and adding to, and understand subtraction as taking apart and taking from.

1. Represent addition and subtraction with objects, fingers, mental images, drawings,[3] sounds (e.g., claps), acting out situations, verbal explanations, expressions, or equations.
2. Solve addition and subtraction word problems, and add and subtract within 10, e.g., by using objects or drawings to represent the problem.
3. Decompose numbers less than or equal to 10 into pairs in more than one way, e.g., by using objects or drawings, and record each decomposition by a drawing or equation (e.g., 5 = 2 + 3 and 5 = 4 + 1).
4. For any number from 1 to 9, find the number that makes 10 when added to the given number, e.g., by using objects or drawings, and record the answer with a drawing or equation.
5. Fluently add and subtract within 5, including zero.

Number and Operations in Base Ten K.NBT

A. Work with numbers 11–19 to gain foundations for place value.

1. Compose and decompose numbers from 11 to 19 into ten ones and some further ones, e.g., by using objects or drawings, and record each composition or decomposition by a drawing or equation (e.g., 18 = 10 + 8); understand that these numbers are composed of ten ones and one, two, three, four, five, six, seven, eight, or nine ones.

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[3] Drawings need not show details, but should show the mathematics in the problem.

Measurement and Data K.MD

A. Describe and compare measurable attributes.

1. Describe measurable attributes of objects, such as length or weight. Describe several measurable attributes of a single object.
2. Directly compare two objects with a measurable attribute in common, to see which object has "more of"/"less of" the attribute, and describe the difference. *For example, directly compare the heights of two children and describe one child as taller/shorter.*

B. Classify objects and count the number of objects in each category.

3. Classify objects into given categories; count the numbers of objects in each category (up to and including 10) and sort the categories by count.

Geometry K.G

A. Identify and describe shapes (squares, circles, triangles, rectangles, hexagons, cubes, cones, cylinders, and spheres).

1. Describe objects in the environment using names of shapes, and describe the relative positions of these objects using terms such as *above*, *below*, *beside*, *in front of*, *behind*, and *next to*.
2. Correctly name shapes regardless of their orientation or overall size.
3. Identify shapes as two-dimensional (lying in a plane, "flat") or three-dimensional ("solid").

B. Analyze, compare, create, and compose shapes.

4. Analyze and compare two- and three-dimensional shapes, in different sizes and orientations, using informal language to describe their similarities, differences, parts (e.g., number of sides and vertices/"corners") and other attributes (e.g., having sides of equal length).
5. Model shapes in the world by building shapes from components (e.g., sticks and clay balls) and drawing shapes.
6. Compose simple shapes to form larger shapes.
   For example, "Can you join these two triangles with full sides touching to make a rectangle?"

Grade 1

Introduction

In grade 1, instructional time should focus on four critical areas: (1) developing understanding of addition, subtraction, and strategies for addition and subtraction within 20; (2) developing understanding of whole number relationships and place value, including grouping in tens and ones; (3) developing understanding of linear measurement and measuring lengths as iterating length units; and (4) reasoning about attributes of, and composing and decomposing geometric shapes.

1. Students develop strategies for adding and subtracting whole numbers based on their prior work with small numbers. They use a variety of models, including discrete objects and length-based models (e.g., cubes connected to form lengths), to model add-to, take-from, put-together, take-apart, and compare situations to develop meaning for the operations of addition and subtraction, and to develop strategies to solve arithmetic problems with these operations. Students understand connections between counting and addition and subtraction (e.g., adding two is the same as counting on two). They use properties of addition to add whole numbers and to create and use increasingly sophisticated strategies based on these properties (e.g., "making tens") to solve addition and subtraction problems within 20. By comparing a variety of solution strategies, children build their understanding of the relationship between addition and subtraction.
2. Students develop, discuss, and use efficient, accurate, and generalizable methods to add within 100 and subtract multiples of 10. They compare whole numbers (at least to 100) to develop an understanding of and solve problems involving their relative sizes. They think of whole numbers between 10 and 100 in terms of tens and ones (especially recognizing the numbers 11 to 19 as composed of a ten and some ones). Through activities that build number sense, they understand the order of the counting numbers and their relative magnitudes.
3. Students develop an understanding of the meaning and processes of measurement, including underlying concepts such as iterating (the mental activity of building up the length of an object with equal-sized units) and the transitivity principle for indirect measurement.[4]
4. Students compose and decompose plane or solid figures (e.g., put two triangles together to make a quadrilateral) and build understanding of part-whole relationships as well as the properties of the original and composite shapes. As they combine shapes, they recognize them from different perspectives and orientations, describe their geometric attributes, and determine how they are alike and different, to develop the background for measurement and for initial understandings of properties such as congruence and symmetry.

The Standards for Mathematical Practice complement the content standards so that students increasingly engage with the subject matter as they grow in mathematical maturity and expertise throughout the elementary, middle, and high school years.

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[4] Students should apply the principle of transitivity of measurement to make indirect comparisons, but they need not use this technical term.

Grade 1 Overview

Operations and Algebraic Thinking

A. Represent and solve problems involving addition and subtraction.
B. Understand and apply properties of operations and the relationship between addition and subtraction.
C. Add and subtract within 20.
D. Work with addition and subtraction equations.

Number and Operations in Base Ten

A. Extend the counting sequence.
B. Understand place value.
C. Use place value understanding and properties of operations to add and subtract.

Measurement and Data

A. Measure lengths indirectly and by iterating length units.
B. Tell and write time.
C. Represent and interpret data.
D. Work with money.

Geometry

A. Reason with shapes and their attributes.

Standards for Mathematical Practice

1. Make sense of problems and persevere in solving them.
2. Reason abstractly and quantitatively.
3. Construct viable arguments and critique the reasoning of others.
4. Model with mathematics.
5. Use appropriate tools strategically.
6. Attend to precision.
7. Look for and make use of structure.
8. Look for and express regularity in repeated reasoning.

Grade 1 Content Standards

Operations and Algebraic Thinking 1.OA

A. Represent and solve problems involving addition and subtraction.

1. Use addition and subtraction within 20 to solve word problems involving situations of adding to, taking from, putting together, taking apart, and comparing, with unknowns in all positions, e.g., by using objects, drawings, and equations (number sentences) with a symbol for the unknown number to represent the problem.[5]
2. Solve word problems that call for addition of three whole numbers whose sum is less than or equal to 20, e.g., by using objects, drawings, and equations with a symbol for the unknown number to represent the problem.

B. Understand and apply properties of operations and the relationship between addition and subtraction.

3. Apply properties of operations to add. [6]

   For example, when adding numbers order does not matter. If 8 + 3 = 11 is known, then 3 + 8 = 11 is also known (Commutative property of addition). To add 2 + 6 + 4, the second two numbers can be added to make a ten, so 2 + 6 + 4 = 2 + 10 = 12 (Associative property of addition). When adding zero to a number, the result is the same number (Identity property of zero for addition).
4. Understand subtraction as an unknown-addend problem.

   For example, subtract 10 – 8 by finding the number that makes 10 when added to 8.

C. Add and subtract within 20.

5. Relate counting to addition and subtraction (e.g., by counting on 2 to add 2).
6. Add and subtract within 20, demonstrating fluency for addition and subtraction within 10. Use mental strategies such as counting on; making 10 (e.g., 8 + 6 = 8 + 2 + 4 = 10 + 4 = 14); decomposing a number leading to a 10 (e.g., 13 – 4 = 13 – 3 – 1 = 10 – 1 = 9); using the relationship between addition and subtraction (e.g., knowing that 8 + 4 = 12, one knows 12 – 8 = 4); and creating equivalent but easier or known sums (e.g., adding 6 + 7 by creating the known equivalent 6 + 6 + 1 = 12 + 1 = 13).

D. Work with addition and subtraction equations.

7. Understand the meaning of the equal sign, and determine if equations involving addition and subtraction are true or false.

   For example, which of the following equations are true and which are false? 6 = 6, 7 = 8 – 1, 5 + 2 = 2 + 5, 4 + 1 = 5 + 2.
8. Determine the unknown whole number in an addition or subtraction equation relating three whole numbers.

   For example, determine the unknown number that makes the equation true in each of the equations 8 + ? = 11, 5 = □– 3, 6 + 6 = □.

Number and Operations in Base Ten 1.NBT

A. Extend the counting sequence.

1. Count to 120, starting at any number less than 120. In this range, read and write numerals and represent a number of objects with a written numeral.

B. Understand place value.

2. Understand that the two digits of a two-digit number represent amounts of tens and ones. Understand the following as special cases:
   a. 10 can be thought of as a bundle of ten ones—called a "ten."
   b. The numbers from 11 to 19 are composed of a ten and one, two, three, four, five, six, seven, eight, or nine ones.

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[5] See Glossary, Table 1.

[6] Students need not use formal terms for these properties.

   - **c.** The numbers 10, 20, 30, 40, 50, 60, 70, 80, 90 refer to one, two, three, four, five, six, seven, eight, or nine tens (and 0 ones).

3. Compare two two-digit numbers based on meanings of the tens and ones digits, recording the results of comparisons with the symbols >, =, and <.

C. Use place value understanding and properties of operations to add and subtract.

4. Add within 100, including adding a two-digit number and a one-digit number, and adding a two-digit number and a multiple of 10, using concrete models or drawings, and strategies based on place value, properties of operations, and/or the relationship between addition and subtraction; relate the strategy to a written method and explain the reasoning used. Understand that in adding two-digit numbers, one adds tens and tens, ones and ones; and sometimes it is necessary to compose a ten.
5. Given a two-digit number, mentally find 10 more or 10 less than the number, without having to count; explain the reasoning used. Identify arithmetic patterns of 10 more and 10 less than using strategies based on place value.
6. Subtract multiples of 10 in the range 10–90 from multiples of 10 in the range 10–90 (positive or zero differences), using concrete models or drawings and strategies based on place value, properties of operations, and/or the relationship between addition and subtraction; relate the strategy to a written method and explain the reasoning used.

Measurement and Data 1.MD

A. Measure lengths indirectly and by iterating length units.

1. Order three objects by length; compare the lengths of two objects indirectly by using a third object.
2. Express the length of an object as a whole number of length units, by laying multiple copies of a shorter object (the length unit) end to end; understand that the length measurement of an object is the number of same-size length units that span it with no gaps or overlaps. *Limit to contexts where the object being measured is spanned by a whole number of length units with no gaps or overlaps.*

B. Tell and write time.

3. Tell and write time in hours and half-hours using analog and digital clocks.

C. Represent and interpret data.

4. Organize, represent, and interpret data with up to three categories; ask and answer questions about the total number of data points, how many in each category, and how many more or less are in one category than in another.

D. Work with money.

5. Identify the values of all U.S. coins and know their comparative values (e.g., a dime is of greater value than a nickel). Find equivalent values (e.g., a nickel is equivalent to five pennies). Use appropriate notation (e.g., 69¢). Use the values of coins in the solutions of problems (up to 100¢).

Geometry 1.G

A. Reason with shapes and their attributes.

1. Distinguish between defining attributes (e.g., triangles are closed and three-sided) versus non-defining attributes (e.g., color, orientation, overall size); build and draw shapes that possess defining attributes.
2. Compose two-dimensional shapes (rectangles, squares, trapezoids, triangles, half-circles, and quarter-circles) or three-dimensional shapes (cubes, right rectangular prisms, right circular cones, and right circular cylinders) to create a composite shape, and compose new shapes from the composite shape.[7]
3. Partition circles and rectangles into two and four equal shares, describe the shares using the words *halves*, *fourths*, and *quarters*, and use the phrases *half of*, *fourth of*, and *quarter of.* Describe the whole as two of, or four of the shares. Understand for these examples that decomposing into more equal shares creates smaller shares.

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[7] Students do not need to learn formal names such as "right rectangular prism."

Grade 2

Introduction

In grade 2, instructional time should focus on four critical areas: (1) extending understanding of base-ten notation; (2) building fluency with addition and subtraction; (3) using standard units of measure; and (4) describing and analyzing shapes.

1. Students extend their understanding of the base-ten system. This includes ideas of counting in fives, tens, and multiples of hundreds, tens, and ones, as well as number relationships involving these units, including comparing. Students understand multi-digit numbers (up to 1,000) written in base-ten notation, recognizing that the digits in each place represent amounts of thousands, hundreds, tens, or ones (e.g., 853 is 8 hundreds + 5 tens + 3 ones).
2. Students use their understanding of addition to develop fluency with addition and subtraction within 100. They solve problems within 1,000 by applying their understanding of models for addition and subtraction, and they develop, discuss, and use efficient, accurate, and generalizable methods to compute sums and differences of whole numbers in base-ten notation, using their understanding of place value and the properties of operations. They select and accurately apply methods that are appropriate for the context and the numbers involved to mentally calculate sums and differences for numbers with only tens or only hundreds.
3. Students recognize the need for standard units of measure (centimeter and inch) and they use rulers and other measurement tools with the understanding that linear measure involves an iteration of units. They recognize that the smaller the unit, the more iterations they need to cover a given length.
4. Students describe and analyze shapes by examining their sides and angles. Students investigate, describe, and reason about decomposing and combining shapes to make other shapes. Through building, drawing, and analyzing two- and three-dimensional shapes, students develop a foundation for understanding area, volume, congruence, similarity, and symmetry in later grades.

The Standards for Mathematical Practice complement the content standards so that students increasingly engage with the subject matter as they grow in mathematical maturity and expertise throughout the elementary, middle, and high school years.

Grade 2 Overview

Operations and Algebraic Thinking

A. Represent and solve problems involving addition and subtraction.
B. Add and subtract within 20.
C. Work with equal groups of objects to gain foundations for multiplication.

Number and Operations in Base Ten

A. Understand place value.
B. Use place value understanding and properties of operations to add and subtract.

Measurement and Data

A. Measure lengths indirectly and by iterating length units.
B. Relate addition and subtraction to length.
C. Work with time and money.
D. Represent and interpret data.

Geometry

A. Reason with shapes and their attributes.

Standards for Mathematical Practice

1. Make sense of problems and persevere in solving them.
2. Reason abstractly and quantitatively.
3. Construct viable arguments and critique the reasoning of others.
4. Model with mathematics.
5. Use appropriate tools strategically.
6. Attend to precision.
7. Look for and make use of structure.
8. Look for and express regularity in repeated reasoning.

Grade 2 Content Standards

Operations and Algebraic Thinking 2.OA

A. Represent and solve problems involving addition and subtraction.

1. Use addition and subtraction within 100 to solve one- and two-step word problems involving situations of adding to, taking from, putting together, taking apart, and comparing, with unknowns in all positions, e.g., by using drawings and equations with a symbol for the unknown number to represent the problem.[8]

B. Add and subtract within 20.

2. Fluently add and subtract within 20 using mental strategies.[9] By end of grade 2, know from memory all sums of two single-digit numbers and related differences.
   For example, the sum 6 + 5 = 11 has related differences of 11 – 5 = 6 and 11 – 6 = 5.

C. Work with equal groups of objects to gain foundations for multiplication.

3. Determine whether a group of objects (up to 20) has an odd or even number of members, e.g., by pairing objects or counting them by 2s; write an equation to express an even number as a sum of two equal addends.
4. Use addition to find the total number of objects arranged in rectangular arrays with up to five rows and up to five columns; write an equation to express the total as a sum of equal addends.

Number and Operations in Base Ten 2.NBT

A. Understand place value.

1. Understand that the three digits of a three-digit number represent amounts of hundreds, tens, and ones; e.g., 706 equals 7 hundreds, 0 tens, and 6 ones. Understand the following as special cases:
   a. 100 can be thought of as a bundle of ten tens—called a "hundred."
   b. The numbers 100, 200, 300, 400, 500, 600, 700, 800, 900 refer to one, two, three, four, five, six, seven, eight, or nine hundreds (and 0 tens and 0 ones).
2. Count within 1,000; skip-count by 5s, 10s, and 100s. Identify patterns in skip counting starting at any number.
3. Read and write numbers to 1,000 using base-ten numerals, number names, and expanded form.
4. Compare two three-digit numbers based on meanings of the hundreds, tens, and ones digits, using >, =, and < symbols to record the results of comparisons.

B. Use place value understanding and properties of operations to add and subtract.

5. Fluently add and subtract within 100 using strategies based on place value, properties of operations, and/or the relationship between addition and subtraction.
6. Add up to four two-digit numbers using strategies based on place value and properties of operations.
7. Add and subtract within 1,000, using concrete models or drawings and strategies based on place value, properties of operations, and/or the relationship between addition and subtraction; relate the strategy to a written method. Understand that in adding or subtracting three-digit numbers, one adds or subtracts hundreds and hundreds, tens and tens, ones and ones; and sometimes it is necessary to compose or decompose tens or hundreds.
8. Mentally add 10 or 100 to a given number 100–900, and mentally subtract 10 or 100 from a given number 100–900.
9. Explain why addition and subtraction strategies work, using place value and the properties of operations.[10]

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[8] See Glossary, Table 1.

[9] Strategies such as counting on; making tens; decomposing a number; using the relationship between addition and subtraction; and creating equivalent but easier or known sums.

[10] Explanations may be supported by drawings or objects.

Measurement and Data 2.MD

A. Measure and estimate lengths in standard units.

1. Measure the length of an object by selecting and using appropriate tools such as rulers, yardsticks, meter sticks, and measuring tapes.
2. Measure the length of an object twice, using length units of different lengths for the two measurements; describe how the two measurements relate to the size of the unit chosen.
3. Estimate lengths using units of inches, feet, centimeters, and meters.
4. Measure to determine how much longer one object is than another, expressing the length difference in terms of a standard length unit.

B. Relate addition and subtraction to length.

5. Use addition and subtraction within 100 to solve word problems involving lengths that are given in the same units, e.g., by using drawings (such as drawings of rulers) and equations with a symbol for the unknown number to represent the problem.
6. Represent whole numbers as lengths from 0 on a number line diagram with equally spaced points corresponding to the numbers 0, 1, 2, …, and represent whole-number sums and differences within 100 on a number line diagram.

C. Work with time and money.

7. Tell and write time from analog and digital clocks to the nearest five minutes, using a.m. and p.m.
    a. Know the relationships of time, including seconds in a minute, minutes in an hour, hours in a day, days in a week; days in a month and a year and approximate number of weeks in a month and weeks in a year.
8. Solve word problems involving dollar bills, quarters, dimes, nickels, and pennies (up to $10), using $ and ¢ symbols appropriately and whole dollar amounts.

    For example, if you have 2 dimes and 3 pennies, how many cents do you have? If you have $3 and 4 quarters, how many dollars or cents do you have? (Students are not expected to use decimal notation.)

D. Represent and interpret data.

9. Generate measurement data by measuring lengths of several objects to the nearest whole unit, or by making repeated measurements of the same object. Organize and record the data on a line plot (dot plot) where the horizontal scale is marked off in whole-number units.
10. Draw a picture graph and a bar graph (with single-unit scale) to represent a data set with up to four categories. Solve simple put-together, take-apart, and compare problems,[11] using information presented in a bar graph.

Geometry 2.G

A. Reason with shapes and their attributes.

1. Recognize and draw shapes having specified attributes, such as a given number of angles or a given number of equal faces.[12] Identify triangles, squares, rectangles, rhombuses, trapezoids, pentagons, hexagons, and cubes.
2. Partition a rectangle into rows and columns of same-size squares and count to find the total number of them.
3. Partition circles and rectangles into two, three, or four equal shares, describe the shares using the words *halves*, *thirds*, *half of*, *a third of*, etc., and describe the whole as two halves, three thirds, four fourths. Recognize that equal shares of identical wholes need not have the same shape.

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[11] See Glossary, Table 1.

[12] Sizes are compared directly or visually, not compared by measuring.

Grade 3

Introduction

In grade 3, instructional time should focus on four critical areas: (1) developing understanding of multiplication and division and strategies for multiplication and division within 100; (2) developing understanding of fractions, especially unit fractions (fractions with numerator 1); (3) developing understanding of the structure of rectangular arrays and of area; and (4) describing and analyzing two-dimensional shapes.

1. Students develop an understanding of the meanings of multiplication and division of whole numbers through activities and problems involving equal-sized groups, arrays, and area models; multiplication is finding an unknown product, and division is finding an unknown factor in these situations. For equal-sized group situations, division can require finding the unknown number of groups or the unknown group size. Students use properties of operations to calculate products of whole numbers, using increasingly sophisticated strategies based on these properties to solve multiplication and division problems involving single-digit factors. By comparing a variety of solution strategies, students learn the relationship between multiplication and division.
2. Students develop an understanding of fractions, beginning with unit fractions. Students view fractions in general as being built out of unit fractions, and they use fractions along with visual fraction models to represent parts of a whole. Students understand that the size of a fractional part is relative to the size of the whole. For example, ½ of the paint in a small bucket could be less paint than ⅓ of the paint in a larger bucket, but ⅓ of a ribbon is longer than ⅕ of the same ribbon because when the ribbon is divided into 3 equal parts, the parts are longer than when the ribbon is divided into 5 equal parts. Students are able to use fractions to represent numbers equal to, less than, and greater than one. They solve problems that involve comparing fractions by using visual fraction models and strategies based on noticing equal numerators or denominators.
3. Students recognize area as an attribute of two-dimensional regions. They measure the area of a shape by finding the total number of same-size units of area required to cover the shape without gaps or overlaps; a square with sides of unit length being the standard unit for measuring area. Students understand that rectangular arrays can be decomposed into identical rows or into identical columns. By decomposing rectangles into rectangular arrays of squares, students connect area to multiplication, and justify using multiplication to determine the area of a rectangle.
4. Students describe, analyze, and compare properties of two-dimensional shapes. They compare and classify shapes by their sides and angles, and connect these with definitions of shapes. Students also relate their fraction work to geometry by expressing the area of part of a shape as a unit fraction of the whole.

The Standards for Mathematical Practice complement the content standards so that students increasingly engage with the subject matter as they grow in mathematical maturity and expertise throughout the elementary, middle, and high school years.

Grade 3 Overview

Operations and Algebraic Thinking

- **A.** Represent and solve problems involving multiplication and division.
- **B.** Understand properties of multiplication and the relationship between multiplication and division.
- **C.** Multiply and divide within 100.
- **D.** Solve problems involving the four operations, and identify and explain patterns in arithmetic.

Number and Operations in Base Ten

- **A.** Use place value understanding and properties of operations to perform multi-digit arithmetic.

Number and Operations—Fractions

- **A.** Develop understanding of fractions as numbers.

Measurement and Data

- **A.** Solve problems involving measurement and estimation of intervals of time, liquid volumes, and masses of objects.
- **B.** Represent and interpret data.
- **C.** Geometric measurement: understand concepts of area and relate area to multiplication and to addition.
- **D.** Geometric measurement: recognize perimeter as an attribute of plane figures and distinguish between linear and area measures.

Geometry

- **A.** Reason with shapes and their attributes.

Standards for Mathematical Practice

1. Make sense of problems and persevere in solving them.
2. Reason abstractly and quantitatively.
3. Construct viable arguments and critique the reasoning of others.
4. Model with mathematics.
5. Use appropriate tools strategically.
6. Attend to precision.
7. Look for and make use of structure.
8. Look for and express regularity in repeated reasoning.

Grade 3 Content Standards

Operations and Algebraic Thinking 3.OA

A. Represent and solve problems involving multiplication and division.

1. Interpret products of whole numbers, e.g., interpret 5×7 as the total number of objects in five groups of seven objects each.

   For example, describe a context in which a total number of objects can be expressed as 5×7.

2. Interpret whole-number quotients of whole numbers, e.g., interpret $56 \div 8$ as the number of objects in each share when 56 objects are partitioned equally into 8 shares, or as a number of shares when 56 objects are partitioned into equal shares of 8 objects each.

   For example, describe a context in which a number of shares or a number of groups can be expressed as $56 \div 8$.

3. Use multiplication and division within 100 to solve word problems in situations involving equal groups, arrays, and measurement quantities, e.g., by using drawings and equations with a symbol for the unknown number to represent the problem.[13]

4. Determine the unknown whole number in a multiplication or division equation relating three whole numbers.

   For example, determine the unknown number that makes the equation true in each of the equations $8 \times ? = 48$, $5 = \square \div 3$, $6 \times 6 = ?$.

B. Understand properties of multiplication and the relationship between multiplication and division.

5. Apply properties of operations to multiply.[14]

   For example, when multiplying numbers order does not matter. If $6 \times 4 = 24$ is known, then $4 \times 6 = 24$ is also known (Commutative property of multiplication); The product $3 \times 5 \times 2$ can be found by $3 \times 5 = 15$ then $15 \times 2 = 30$, or by $5 \times 2 = 10$ then $3 \times 10 = 30$ (Associative property of multiplication); When multiplying two numbers either number can be decomposed and multiplied; one can find 8 x 7 by knowing that $7 = 5 + 2$ and that $8 \times 5 = 40$ and $8 \times 2 = 16$, resulting in $8 \times (5 + 2) = (8 \times 5) + (8 \times 2) = 40 + 16 = 56$ (Distributive property); When a number is multiplied by 1 the result is the same number (Identity property of 1 for multiplication).

6. Understand division as an unknown-factor problem.

   For example, find $32 \div 8$ by finding the number that makes 32 when multiplied by 8.

C. Multiply and divide within 100.

7. Fluently multiply and divide within 100, using strategies such as the relationship between multiplication and division (e.g., knowing that $8 \times 5 = 40$, one knows $40 \div 5 = 8$) or properties of operations. By the end of grade 3, know from memory all products of two single-digit numbers and related division facts.

   For example, the product $4 \times 7 = 28$ has related division facts $28 \div 7 = 4$ and $28 \div 4 = 7$.

D. Solve problems involving the four operations, and identify and explain patterns in arithmetic.

8. Solve two-step word problems using the four operations for problems posed with whole numbers and having whole number answers. Represent these problems using equations with a letter standing for the unknown quantity. Assess the reasonableness of answers using mental computation and estimation strategies, including rounding.[15]

9. Identify arithmetic patterns (including patterns in the addition table or multiplication table) and explain them using properties of operations.

   For example, observe that 4 times a number is always even, and explain why 4 times a number can be decomposed into two equal addends.

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[13] See Glossary, Table 2.

[14] Students need not use formal terms for these properties. Students are not expected to use distributive notation.

[15] Students should know how to perform operations in the conventional order when there are no parentheses to specify a particular order (Order of Operations).

Number and Operations in Base Ten 3.NBT

A. Use place value understanding and properties of operations to perform multi-digit arithmetic. [16]

1. Use place value understanding to round whole numbers to the nearest 10 or 100.
2. Fluently add and subtract within 1,000 using strategies and algorithms based on place value, properties of operations, and/or the relationship between addition and subtraction.
3. Multiply one-digit whole numbers by multiples of 10 in the range 10–90 (e.g., 9 × 80, 5 × 60) using strategies based on place value and properties of operations.

Number and Operations—Fractions 3.NF

A. Develop understanding of fractions as numbers for fractions with denominators 2, 3, 4, 6, and 8.

1. Understand a fraction $^1/_b$ as the quantity formed by 1 part when a whole (a single unit) is partitioned into *b* equal parts; understand a fraction $^a/_b$ as the quantity formed by *a* parts of size $^1/_b$.
2. Understand a fraction as a number on the number line; represent fractions on a number line diagram.
    - **a.** Represent a unit fraction, $^1/_b$, on a number line diagram by defining the interval from 0 to 1 as the whole and partitioning it into *b* equal parts. Recognize that each part has size $^1/_b$ and that the fraction $^1/_b$ is located $^1/_b$ of a whole unit from 0 on the number line.
    - **b.** Represent a fraction $^a/_b$ on a number line diagram by marking off *a* lengths $^1/_b$ from 0. Recognize that the resulting interval has size $^a/_b$ and that its endpoint locates the number $^a/_b$ on the number line.
3. Explain equivalence of fractions in special cases, and compare fractions by reasoning about their size.
    - **a.** Understand two fractions as equivalent (equal) if they are the same size, or the same point on a number line.
    - **b.** Recognize and generate simple equivalent fractions, e.g., $^1/_2 = {}^2/_4$, $^4/_6 = {}^2/_3$. Explain why the fractions are equivalent, e.g., by using a visual fraction model.
    - **c.** Express whole numbers as fractions, and recognize fractions that are equivalent to whole numbers.

For example, express 3 in the form 3 = $^3/_1$; recognize that $^6/_1$ = 6; locate $^4/_4$ and 1 at the same point of a number line diagram.

    - **d.** Compare two fractions with the same numerator or the same denominator by reasoning about their size. Recognize that comparisons are valid only when the two fractions refer to the same whole. Record the results of comparisons with the symbols >, =, or <, and justify the conclusions, e.g., by using a visual fraction model.

Measurement and Data 3.MD

A. Solve problems involving measurement and estimation of intervals of time, liquid volumes, and masses of objects.

1. Tell and write time to the nearest minute and measure time intervals in minutes. Solve word problems involving addition and subtraction of time intervals in minutes, e.g., by representing the problem on a number line diagram.
2. Measure and estimate liquid volumes and masses of objects using standard metric units of grams (g), kilograms (kg), and liters (*l*).[17] Add, subtract, multiply, or divide to solve one-step word problems involving masses or volumes that are given in the same metric units, e.g., by using drawings (such as a beaker with a measurement scale) to represent the problem.[18]

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[16] A range of algorithms may be used.

[17] Excludes compound units such as cm^3 and finding the geometric volume of a container.

[18] Excludes multiplicative comparison problems (problems involving notions of "times as much"; see Glossary, Table 2).

B. Represent and interpret data.

3. Draw a scaled picture graph and a scaled bar graph to represent a data set with several categories. Solve one- and two-step "how many more" and "how many less" problems using information presented in scaled bar graphs.

   For example, draw a bar graph in which each square in the bar graph might represent five pets.

4. Generate measurement data by measuring lengths of objects using rulers marked with halves and fourths of an inch. Record and show the data by making a line plot (dot plot), where the horizontal scale is marked off in appropriate units—whole numbers, halves, or fourths. *(See Glossary for example.)*

C. Geometric measurement: understand concepts of area and relate area to multiplication and to addition.

5. Recognize area as an attribute of plane figures and understand concepts of area measurement.
   - **a.** A square with side length of one unit, called "a unit square," is said to have "one square unit" of area, and can be used to measure area.
   - **b.** A plane figure which can be covered without gaps or overlaps by n unit squares is said to have an area of n square units.
6. Measure areas by counting unit squares (square cm, square m, square in., square ft., and non-standard units).
7. Relate area to the operations of multiplication and addition.
   - **a.** Find the area of a rectangle with whole-number side lengths by tiling it, and show that the area is the same as would be found by multiplying the side lengths.
   - **b.** Multiply side lengths to find areas of rectangles with whole-number side lengths in the context of solving real-world and mathematical problems, and represent whole-number products as rectangular areas in mathematical reasoning.
   - **c.** Use tiling to show in a concrete case that the area of a rectangle with whole-number side lengths a and $b + c$ is the sum of $a \times b$ and $a \times c$. Use area models to represent the distributive property in mathematical reasoning.
   - **d.** Recognize area as additive. Find areas of rectilinear figures by decomposing them into non-overlapping rectangles and adding the areas of the non-overlapping parts, applying this technique to solve real-world problems.

D. Geometric measurement: recognize perimeter as an attribute of plane figures and distinguish between linear and area measures.

8. Solve real-world and mathematical problems involving perimeters of polygons, including finding the perimeter given the side lengths, finding an unknown side length, and exhibiting rectangles with the same perimeter and different areas or with the same area and different perimeters.

Geometry 3.G

A. Reason with shapes and their attributes.

1. Understand that shapes in different categories (e.g., rhombuses, rectangles, and others) may share attributes (e.g., having four sides), and that the shared attributes can define a larger category (e.g., quadrilaterals). Compare and classify shapes by their sides and angles (right angle/non-right angle). Recognize rhombuses, rectangles, squares, and trapezoids as examples of quadrilaterals, and draw examples of quadrilaterals that do not belong to any of these subcategories.
2. Partition shapes into parts with equal areas. Express the area of each part as a unit fraction of the whole.

   For example, partition a shape into four parts with equal areas and describe the area of each part as ¼ of the area of the shape.

Grade 4

Introduction

In grade 4, instructional time should focus on three critical areas: (1) developing understanding and fluency with multi-digit multiplication, and developing understanding of dividing to find quotients involving multi-digit dividends; (2) developing an understanding of fraction equivalence, addition and subtraction of fractions with like denominators, and multiplication of fractions by whole numbers; (3) and understanding that geometric figures can be analyzed and classified based on their properties, such as having parallel sides, perpendicular sides, particular angle measures, and symmetry.

1. Students generalize their understanding of place value to 1,000,000, understanding the relative sizes of numbers in each place. They apply their understanding of models for multiplication (equal-sized groups, arrays, area models), place value, and properties of operations, in particular the distributive property, as they develop, discuss, and use efficient, accurate, and generalizable methods to compute products of multi-digit whole numbers. Depending on the numbers and the context, they select and accurately apply appropriate methods to estimate or mentally calculate products. They develop fluency with efficient procedures for multiplying whole numbers; understand and explain why the procedures work based on place value and properties of operations; and use them to solve problems. Students apply their understanding of models for division, place value, properties of operations, and the relationship of division to multiplication as they develop, discuss, and use efficient, accurate, and generalizable procedures to find quotients involving multi-digit dividends. They select and accurately apply appropriate methods to estimate and mentally calculate quotients, and interpret remainders based upon the context.
2. Students develop understanding of fraction equivalence and operations with fractions. They recognize that two different fractions can be equal (e.g., $^{15}/_{9} = {}^{5}/_{3}$), and they develop methods for generating and recognizing equivalent fractions. Students extend previous understandings about how fractions are built from unit fractions, composing fractions from unit fractions, decomposing fractions into unit fractions, and using the meaning of fractions and the meaning of multiplication to multiply a fraction by a whole number.
3. Students describe, analyze, compare, and classify two-dimensional shapes. Through building, drawing, and analyzing two-dimensional shapes, students deepen their understanding of properties of two-dimensional objects and the use of them to solve problems involving symmetry.

The Standards for Mathematical Practice complement the content standards so that students increasingly engage with the subject matter as they grow in mathematical maturity and expertise throughout the elementary, middle, and high school years.

Grade 4 Overview

Operations and Algebraic Thinking

A. Use the four operations with whole numbers to solve problems.
B. Gain familiarity with factors and multiples.
C. Generate and analyze patterns.

Number and Operations in Base Ten

A. Generalize place value understanding for multi-digit whole numbers less than or equal to 1,000,000.
B. Use place value understanding and properties of operations to perform multi-digit arithmetic on whole numbers less than or equal to 1,000,000.

Number and Operations—Fractions

A. Extend understanding of fraction equivalence and ordering for fractions ordering for fractions with denominators 2, 3, 4, 5, 6, 8, 10, 12, and 100.
B. Build fractions from unit fractions by applying and extending previous understandings of operations on whole numbers for fractions with denominators 2, 3, 4, 5, 6, 8, 10, 12, and 100.
C. Understand decimal notation for fractions, and compare decimal fractions.

Standards for Mathematical Practice

1. Make sense of problems and persevere in solving them.
2. Reason abstractly and quantitatively.
3. Construct viable arguments and critique the reasoning of others.
4. Model with mathematics.
5. Use appropriate tools strategically.
6. Attend to precision.
7. Look for and make use of structure.
8. Look for and express regularity in repeated reasoning.

Measurement and Data

A. Solve problems involving measurement and conversion of measurements from a larger unit to a smaller unit.
B. Represent and interpret data.
C. Geometric measurement: Understand concepts of angle and measure angles.

Geometry

A. Draw and identify lines and angles, and classify shapes by properties of their lines and angles.

Grade 4 Content Standards

Operations and Algebraic Thinking 4.OA

A. Use the four operations with whole numbers to solve problems.

1. Interpret a multiplication equation as a comparison, e.g., interpret $35 = 5 \times 7$ as a statement that 35 is 5 times as many as 7 and 7 times as many as 5. Represent verbal statements of multiplicative comparisons as multiplication equations.
2. Multiply or divide to solve word problems involving multiplicative comparison, e.g., by using drawings and equations with a symbol for the unknown number to represent the problem, distinguishing multiplicative comparison from additive comparison.[19]
3. Solve multi-step word problems posed with whole numbers and having whole-number answers using the four operations, including problems in which remainders must be interpreted. Represent these problems using equations with a letter standing for the unknown quantity. Assess the reasonableness of answers using mental computation and estimation strategies including rounding.
    a. Know multiplication facts and related division facts through 12 x 12.

B. Gain familiarity with factors and multiples.

4. Find all factor pairs for a whole number in the range 1–100. Recognize that a whole number is a multiple of each of its factors. Determine whether a given whole number in the range 1–100 is a multiple of a given one-digit number. Determine whether a given whole number in the range 1–100 is prime or composite.

C. Generate and analyze patterns.

5. Generate a number or shape pattern that follows a given rule. Identify apparent features of the pattern that were not explicit in the rule itself.

    For example, given the rule "Add 3" and the starting number 1, generate terms in the resulting sequence and observe that the terms appear to alternate between odd and even numbers. Explain informally why the numbers will continue to alternate in this way.

Number and Operations in Base Ten 4.NBT

A. Generalize place value understanding for multi-digit whole numbers less than or equal to 1,000,000.

1. Recognize that in a multi-digit whole number, a digit in any place represents 10 times as much as it represents in the place to its right.

    For example, recognize that $700 \div 70 = 10$ ***by applying concepts of place value and division.***

2. Read and write multi-digit whole numbers using base-ten numerals, number names, and expanded form. Compare two multi-digit numbers based on meanings of the digits in each place, using >, =, and < symbols to record the results of comparisons.
3. Use place value understanding to round multi-digit whole numbers to any place.

B. Use place value understanding and properties of operations to perform multi-digit arithmetic on whole numbers less than or equal to 1,000,000.

4. Fluently add and subtract multi-digit whole numbers using the standard algorithm.
5. Multiply a whole number of up to four digits by a one-digit whole number, and multiply two two-digit numbers, using strategies based on place value and the properties of operations. Illustrate and explain the calculation by using equations, rectangular arrays, and/or area models.
6. Find whole-number quotients and remainders with up to four-digit dividends and one-digit divisors, using strategies based on place value, the properties of operations, and/or the relationship between multiplication and division. Illustrate and explain the calculation by using equations, rectangular arrays, and/or area models.

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[19] See Glossary, Table 2.

Number and Operations—Fractions 4.NF

A. Extend understanding of fraction equivalence and ordering for fractions with denominators 2, 3, 4, 5, 6, 8, 10, 12, and 100.

1. Explain why a fraction $^{a}/_{b}$ is equivalent to a fraction $^{(n \times a)}/_{(n \times b)}$ by using visual fraction models, with attention to how the numbers and sizes of the parts differ even though the two fractions themselves are the same size. Use this principle to recognize and generate equivalent fractions, including fractions greater than 1.
2. Compare two fractions with different numerators and different denominators, e.g., by creating common denominators or numerators, or by comparing to a benchmark fraction such as $^{1}/_{2}$. Recognize that comparisons are valid only when the two fractions refer to the same whole. Record the results of comparisons with symbols >, =, or <, and justify the conclusions, e.g., by using a visual fraction model.

B. Build fractions from unit fractions by applying and extending previous understandings of operations on whole numbers for fractions with denominators 2, 3, 4, 5, 6, 8, 10, 12, and 100.

3. Understand a fraction $^{a}/_{b}$ with $a > 1$ as a sum of fractions $^{1}/_{b}$.
   - **a.** Understand addition and subtraction of fractions as joining and separating parts referring to the same whole. (The whole can be a set of objects.)
   - **b.** Decompose a fraction into a sum of fractions with the same denominator in more than one way, recording each decomposition by an equation. Justify decompositions, e.g., by using drawings or visual fraction models. *Examples:* $^{3}/_{8} = {}^{1}/_{8} + {}^{1}/_{8} + {}^{1}/_{8}$; $^{3}/_{8} = {}^{1}/_{8} + {}^{2}/_{8}$; $2\,^{1}/_{8} = 1 + 1 + {}^{1}/_{8} = {}^{8}/_{8} + {}^{8}/_{8} + {}^{1}/_{8}$.
   - **c.** Add and subtract mixed numbers with like denominators, e.g., by replacing each mixed number with an equivalent fraction, and/or by using properties of operations and the relationship between addition and subtraction.
   - **d.** Solve word problems involving addition and subtraction of fractions referring to the same whole and having like denominators, e.g., by using drawings or visual fraction models and equations to represent the problem.
4. Apply and extend previous understandings of multiplication to multiply a fraction by a whole number.
   - **a.** Understand a fraction $^{a}/_{b}$ as a multiple of $^{1}/_{b}$.

     For example, use a visual fraction model to represent $^{5}/_{4}$ as the product $5 \times (^{1}/_{4})$, recording the conclusion by the equation $^{5}/_{4} = 5 \times (^{1}/_{4})$.
   - **b.** Understand a multiple of $^{a}/_{b}$ as a multiple of $^{1}/_{b}$, and use this understanding to multiply a fraction by a whole number.

     For example, use a visual fraction model to express $3 \times (^{2}/_{5})$ as $6 \times (^{1}/_{5})$, recognizing this product as $^{6}/_{5}$. (In general, $n \times (^{a}/_{b}) = {}^{(n \times a)}/_{b}$.)
   - **c.** Solve word problems involving multiplication of a fraction by a whole number, e.g., by using visual fraction models and equations to represent the problem.

     For example, if each person at a party will eat $^{3}/_{8}$ of a pound of roast beef, and there will be 5 people at the party, how many pounds of roast beef will be needed? Between what two whole numbers does your answer lie?

C. Understand decimal notation for fractions, and compare decimal fractions.

5. Express a fraction with denominator 10 as an equivalent fraction with denominator 100, and use this technique to add two fractions with respective denominators 10 and 100.[20]

   For example, express $^{3}/_{10}$ as $^{30}/_{100}$, and add $^{3}/_{10} + {}^{4}/_{100} = {}^{34}/_{100}$.
6. Use decimal notation to represent fractions with denominators 10 or 100.

   For example, rewrite 0.62 as $^{62}/_{100}$; describe a length as 0.62 meters; locate 0.62 on a number line diagram.

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[20] Students who can generate equivalent fractions can develop strategies for adding fractions with unlike denominators in general. But addition and subtraction with unlike denominators in general is not a requirement at this grade.

7. Compare two decimals to hundredths by reasoning about their size. Recognize that comparisons are valid only when the two decimals refer to the same whole. Record the results of comparisons with the symbols >, =, or <, and justify the conclusions, e.g., by using a visual model.

Measurement and Data 4.MD

A. Solve problems involving measurement and conversion of measurements from a larger unit to a smaller unit.

1. Know relative sizes of measurement units within one system of units, including km, m, cm; kg, g; lb, oz.; l, ml; hr, min, sec. Within a single system of measurement, express measurements in a larger unit in terms of a smaller unit. Record measurement equivalents in a two-column table.
   For example, know that 1 ft. is 12 times as long as 1 in. Express the length of a 4 ft. snake as 48 in. Generate a conversion table for feet and inches listing the number pairs (1, 12), (2, 24), (3, 36), …
2. Use the four operations to solve word problems involving distances, intervals of time, liquid volumes, masses of objects, and money, including problems involving simple fractions or decimals, and problems that require expressing measurements given in a larger unit in terms of a smaller unit. Represent measurement quantities using diagrams such as number line diagrams that feature a measurement scale.
3. Apply the area and perimeter formulas for rectangles in real-world and mathematical problems.
   For example, find the width of a rectangular room given the area of the flooring and the length, by viewing the area formula as a multiplication equation with an unknown factor. (Note: When finding areas of rectangular regions answers will be in square units. For example, the area of a 1 cm x 1 cm rectangular region will be 1 square centimeter (1 cm^2, students are not expected to use this notation.) When finding the perimeter of a rectangular region answers will be in linear units. For example, the perimeter of the region is: 1cm + 1cm + 1cm +1cm = 4 cm or 2(1cm) + 2(1cm) = 4 cm).

B. Represent and interpret data.

4. Make a line plot (dot plot) representation to display a data set of measurements in fractions of a unit ($^1/_2$, $^1/_4$, $^1/_8$). Solve problems involving addition and subtraction of fractions by using information presented in line plots (dot plots).
   For example, from a line plot (dot plot) find and interpret the difference in length between the longest and shortest specimens in an insect collection.

C. Geometric measurement: Understand concepts of angle and measure angles.

5. Recognize angles as geometric shapes that are formed wherever two rays share a common endpoint, and understand concepts of angle measurement:
   a. An angle is measured with reference to a circle with its center at the common endpoint of the rays, by considering the fraction of the circular arc between the points where the two rays intersect the circle. An angle that turns through $^1/_{360}$ of a circle is called a "one-degree angle," and can be used to measure angles.
   b. An angle that turns through *n* one-degree angles is said to have an angle measure of *n* degrees.
6. Measure angles in whole-number degrees using a protractor. Sketch angles of specified measure.
7. Recognize angle measure as additive. When an angle is decomposed into non-overlapping parts, the angle measure of the whole is the sum of the angle measures of the parts. Solve addition and subtraction problems to find unknown angles on a diagram in real-world and mathematical problems, e.g., by using an equation with a symbol for the unknown angle measure.

Geometry 4.G

A. Draw and identify lines and angles, and classify shapes by properties of their lines and angles.

1. Draw points, lines, line segments, rays, angles (right, acute, obtuse), and perpendicular and parallel lines. Identify these in two-dimensional figures.

2. Classify two-dimensional figures based on the presence or absence of parallel or perpendicular lines, or the presence or absence of angles of a specified size. Recognize right triangles as a category, and identify right triangles.
3. Recognize a line of symmetry for a two-dimensional figure as a line across the figure such that the figure can be folded along the line into matching parts. Identify line-symmetric figures and draw lines of symmetry.

Grade 5

Introduction

In grade 5, instructional time should focus on four critical areas: (1) developing fluency with addition and subtraction of fractions, and developing understanding of the multiplication of fractions and of division of fractions in limited cases (unit fractions divided by whole numbers and whole numbers divided by unit fractions); (2) extending division to 2-digit divisors, integrating decimal fractions into the place value system and developing understanding of operations with decimals to hundredths, and developing fluency with whole number and decimal operations; and (3) developing understanding of measurement systems and determining volumes to solve problems; and (4) solving problems using the coordinate plane

1. Students apply their understanding of fractions and fraction models to represent the addition and subtraction of fractions with unlike denominators as equivalent calculations with like denominators. They develop fluency in calculating sums and differences of fractions, and make reasonable estimates of them. Students also use the meaning of fractions, of multiplication and division, and the relationship between multiplication and division to understand and explain why the procedures for multiplying and dividing fractions make sense. (Note: this is limited to the case of dividing unit fractions by whole numbers and whole numbers by unit fractions.)
2. Students develop understanding of why division procedures work based on the meaning of base-ten numerals and properties of operations. They finalize fluency with multi-digit multiplication, and division. They apply their understandings of models for decimals, decimal notation, and properties of operations to add and subtract decimals to hundredths. They develop fluency in these computations and make reasonable estimates of their results. Students use the relationship between decimals and fractions, as well as the relationship between finite decimals and whole numbers (i.e., a finite decimal multiplied by an appropriate power of 10 is a whole number), to understand and explain why the procedures for multiplying and dividing finite decimals make sense. They compute products and quotients of decimals to hundredths efficiently and accurately.
3. Students convert among different-sized measurement units within a given measurement system allowing for efficient and accurate problem solving with multi-step real-world problems as they progress in their understanding of scientific concepts and calculations. Students recognize volume as an attribute of three-dimensional space. They understand that volume can be measured by finding the total number of same-size units of volume required to fill the space without gaps or overlaps. They understand that a 1-unit by 1-unit by 1-unit cube is the standard unit for measuring volume. They select appropriate units, strategies, and tools for solving problems that involve estimating and measuring volume. They decompose three-dimensional shapes and find volumes of right rectangular prisms by viewing them as decomposed into layers of arrays of cubes. They measure necessary attributes of shapes in order to determine volumes to solve real-world and mathematical problems.
4. Students learn to interpret the components of a rectangular coordinate system as lines and understand the precision of location that these lines require. Students learn to apply their knowledge of number and length to the order and distance relationships of a coordinate grid and to coordinate this across two dimensions. Students solve mathematical and real world problems using coordinates.

The Standards for Mathematical Practice complement the content standards so that students increasingly engage with the subject matter as they grow in mathematical maturity and expertise throughout the elementary, middle, and high school years.

Grade 5 Overview

Operations and Algebraic Thinking

A. Write and interpret numerical expressions.
B. Analyze patterns and relationships.

Number and Operations in Base Ten

A. Understand the place value system.
B. Perform operations with multi-digit whole numbers and with decimals to hundredths.

Number and Operations—Fractions

A. Use equivalent fractions as a strategy to add and subtract fractions.
B. Apply and extend previous understandings of multiplication and division to multiply and divide fractions.

Measurement and Data

A. Convert like measurement units within a given measurement system.
B. Represent and interpret data.
C. Geometric measurement: Understand concepts of volume and relate volume to multiplication and to addition.

Geometry

A. Graph points on the coordinate plane to solve real-world and mathematical problems.
B. Classify two-dimensional figures into categories based on their properties.

Standards for Mathematical Practice

1. Make sense of problems and persevere in solving them.
2. Reason abstractly and quantitatively.
3. Construct viable arguments and critique the reasoning of others.
4. Model with mathematics.
5. Use appropriate tools strategically.
6. Attend to precision.
7. Look for and make use of structure.
8. Look for and express regularity in repeated reasoning.

Grade 5 Content Standards

Operations and Algebraic Thinking 5.OA

A. Write and interpret numerical expressions.

1. Use parentheses, brackets, or braces in numerical expressions, and evaluate expressions with these symbols, e.g.,(6 x 30) + (6 x $\frac{1}{2}$).
2. Write simple expressions that record calculations with numbers, and interpret numerical expressions without evaluating them.

   For example, express the calculation "Add 8 and 7, then multiply by 2" as 2 × (8 + 7). Recognize that 3 × (18932 + 921) is three times as large as 18932 + 921, without having to calculate the indicated sum or product.

B. Analyze patterns and relationships.

3. Generate two numerical patterns using two given rules. Identify apparent relationships between corresponding terms. Form ordered pairs consisting of corresponding terms from the two patterns, and graph the ordered pairs on a coordinate plane.

   For example, given the rule "Add 3" and the starting number 0, and given the rule "Add 6" and the starting number 0, generate terms in the resulting sequences, and observe that the terms in one sequence are twice the corresponding terms in the other sequence. Explain informally why this is so.

Number and Operations in Base Ten 5.NBT

A. Understand the place value system.

1. Recognize that in a multi-digit number, including decimals, a digit in any place represents 10 times as much as it represents in the place to its right and $\frac{1}{10}$ of what it represents in the place to its left.
2. Explain patterns in the number of zeros of the product when multiplying a number by powers of 10, and explain patterns in the placement of the decimal point when a decimal is multiplied or divided by a power of 10. Use whole-number exponents to denote powers of 10.
3. Read, write, and compare decimals to thousandths.
   - **a.** Read and write decimals to thousandths using base-ten numerals, number names, and expanded form, e.g., $347.392 = 3 \times 100 + 4 \times 10 + 7 \times 1 + 3 \times (\frac{1}{10}) + 9 \times (\frac{1}{100}) + 2 \times (\frac{1}{1000})$.
   - **b.** Compare two decimals to thousandths based on meanings of the digits in each place, using >, =, and < symbols to record the results of comparisons.
4. Use place value understanding to round decimals to any place.

B. Perform operations with multi-digit whole numbers and with decimals to hundredths.

5. Fluently multiply multi-digit whole numbers. (Include two-digit x four-digit numbers and, three-digit x three-digit numbers) using the standard algorithm.
6. Find whole-number quotients of whole numbers with up to four-digit dividends and two-digit divisors, using strategies based on place value, the properties of operations, and/or the relationship between multiplication and division. Illustrate and explain the calculation by using equations, rectangular arrays, and/or area models.
7. Add, subtract, multiply, and divide decimals to hundredths, using concrete models or drawings and strategies based on place value, properties of operations, and/or the relationship between addition and subtraction and between multiplication and division; relate the strategy to a written method and explain the reasoning used.

Number and Operations—Fractions 5.NF

A. Use equivalent fractions as a strategy to add and subtract fractions.

1. Add and subtract fractions with unlike denominators (including mixed numbers) by replacing given fractions with equivalent fractions in such a way as to produce an equivalent sum or difference of fractions with like denominators.

For example, $^{2}/_{3} + {}^{5}/_{4} = {}^{8}/_{12} + {}^{15}/_{12} = {}^{23}/_{12}$. (In general, $^{a}/_{b} + {}^{c}/_{d} = {}^{(ad + bc)}/_{bd}$.)

2. Solve word problems involving addition and subtraction of fractions referring to the same whole (the whole can be a set of objects), including cases of unlike denominators, e.g., by using visual fraction models or equations to represent the problem. Use benchmark fractions and number sense of fractions to estimate mentally and assess the reasonableness of answers.

   For example, recognize an incorrect result $^{2}/_{5} + {}^{1}/_{2} = {}^{3}/_{7}$, by observing that $^{3}/_{7} < {}^{1}/_{2}$.

B. Apply and extend previous understandings of multiplication and division to multiply and divide fractions.

3. Interpret a fraction as division of the numerator by the denominator ($^{a}/_{b} = a \div b$). Solve word problems involving division of whole numbers leading to answers in the form of fractions or mixed numbers, e.g., by using visual fraction models or equations to represent the problem.

   For example, interpret ¾ as the result of dividing 3 by 4, noting that ¾ multiplied by 4 equals 3, and that when three wholes are shared equally among four people each person has a share of size ¾. If nine people want to share a 50-pound sack of rice equally by weight, how many pounds of rice should each person get? Between what two whole numbers does your answer lie?

4. Apply and extend previous understandings of multiplication to multiply a fraction or whole number by a fraction.
   - **a.** Interpret the product $(^{a}/_{b}) \times q$ as a parts of a partition of q into b equal parts; equivalently, as the result of a sequence of operations $a \times q \div b$.

   For example, use a visual fraction model and/or area model to show $(^{2}/_{3}) \times 4 = {}^{8}/_{3}$, and create a story context for this equation. Do the same with $(^{2}/_{3}) \times (^{4}/_{5}) = {}^{8}/_{15}$. (In general, $(^{a}/_{b}) \times (^{c}/_{d}) = {}^{ac}/_{bd}$.)

   - **b.** Find the area of a rectangle with fractional side lengths by tiling it with unit squares of the appropriate unit fraction side lengths, and show that the area is the same as would be found by multiplying the side lengths. Multiply fractional side lengths to find areas of rectangles, and represent fraction products as rectangular areas.

5. Interpret multiplication as scaling (resizing), by:
   - **a.** Comparing the size of a product to the size of one factor on the basis of the size of the other factor, without performing the indicated multiplication.

   For example, without multiplying tell which number is greater: 225 or ¾ x 225; $^{11}/_{50}$ or $^{3}/_{2}$ x $^{11}/_{50}$?

   - **b.** Explaining why multiplying a given number by a fraction greater than 1 results in a product greater than the given number (recognizing multiplication by whole numbers greater than 1 as a familiar case); explaining why multiplying a given number by a fraction less than 1 results in a product smaller than the given number; and relating the principle of fraction equivalence $^{a}/_{b} = (n \times a)/(n \times b)$ to the effect of multiplying $^{a}/_{b}$ by 1.

6. Solve real-world problems involving multiplication of fractions and mixed numbers, e.g., by using visual fraction models or equations to represent the problem.

7. Apply and extend previous understandings of division to divide unit fractions by whole numbers and whole numbers by unit fractions.[21]
   - **a.** Interpret division of a unit fraction by a non-zero whole number, and compute such quotients.

   For example, create a story context for $(^{1}/_{3}) \div 4$, and use a visual fraction model to show the quotient. Use the relationship between multiplication and division to explain that $(^{1}/_{3}) \div 4 = {}^{1}/_{12}$ because $(^{1}/_{12}) \times 4 = {}^{1}/_{3}$.

   - **b.** Interpret division of a whole number by a unit fraction, and compute such quotients.

   For example, create a story context for $4 \div (^{1}/_{5})$, and use a visual fraction model to show the quotient. Use the relationship between multiplication and division to explain that $4 \div (^{1}/_{5}) = 20$ because $20 \times (^{1}/_{5}) = 4$.

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[21] Students able to multiply fractions in general can develop strategies to divide fractions in general, by reasoning about the relationship between multiplication and division. But division of a fraction by a fraction is not a requirement at this grade.

   - **c.** Solve real-world problems involving division of unit fractions by non-zero whole numbers and division of whole numbers by unit fractions, e.g., by using visual fraction models and equations to represent the problem.

   For example, how much chocolate will each person get if three people share ½ lb. of chocolate equally? How many ⅓-cup servings are in two cups of raisins?

Measurement and Data 5.MD

A. Convert like measurement units within a given measurement system.

1. Convert among different-sized standard measurement units within a given measurement system (e.g., convert 5 cm to 0.05 m), and use these conversions in solving multi-step, real-world problems.

B. Represent and interpret data.

2. Make a line plot (dot plot) to display a data set of measurements in fractions of a unit. Use operations on fractions for this grade to solve problems involving information presented in line plot (dot plot).

   For example, given different measurements of liquid in identical beakers, find the amount of liquid each beaker would contain if the total amount in all the beakers were redistributed equally.

C. Geometric measurement: Understand concepts of volume and relate volume to multiplication and to addition.

3. Recognize volume as an attribute of solid figures and understand concepts of volume measurement.
   - **a.** A cube with side length 1 unit, called a "unit cube," is said to have "one cubic unit" of volume, and can be used to measure volume.
   - **b.** A solid figure which can be packed without gaps or overlaps using *n* unit cubes is said to have a volume of *n* cubic units.
4. Measure volumes by counting unit cubes, using cubic cm, cubic in., cubic ft., and non-standard units.
5. Relate volume to the operations of multiplication and addition and solve real-world and mathematical problems involving volume.
   - **a.** Find the volume of a right rectangular prism with whole-number edge lengths by packing it with unit cubes, and show that the volume is the same as would be found by multiplying the edge lengths, equivalently by multiplying the height by the area of the base. Represent threefold whole-number products as volumes, e.g., to represent the associative property of multiplication.
   - **b.** Apply the formula $V = l \times w \times h$ and $V = B \times h$ (where B stands for the area of the base) for rectangular prisms to find volumes of right rectangular prisms with whole-number edge lengths in the context of solving real-world and mathematical problems.
   - **c.** Recognize volume as additive. Find volumes of solid figures composed of two non-overlapping right rectangular prisms by adding the volumes of the non-overlapping parts, applying this technique to solve real-world problems.

Geometry 5.G

A. Graph points on the coordinate plane to solve real-world and mathematical problems.

1. Use a pair of perpendicular number lines, called axes, to define a coordinate system, with the intersection of the lines (the origin) arranged to coincide with the zero on each line and a given point in the plane located by using an ordered pair of numbers, called its coordinates. Understand that the first number indicates how far to travel from the origin in the direction of one axis, and the second number indicates how far to travel in the direction of the second axis, with the convention that the names of the two axes and the coordinates correspond (e.g., *x*-axis and *x*-coordinate, *y*-axis and *y*-coordinate).
2. Represent real-world and mathematical problems by graphing points in the first quadrant of the coordinate plane, and interpret coordinate values of points in the context of the situation.

B. Classify two-dimensional figures into categories based on their properties.

3. Understand that attributes belonging to a category of two-dimensional figures also belong to all subcategories of that category.

For example, all rectangles have four right angles and squares are rectangles, so all squares have four right angles.

4. Classify two-dimensional figures in a hierarchy based on properties.

For example, all rectangles are parallelograms because they are all quadrilaterals with two pairs of opposite sides parallel.

Grade 6

Introduction

In grade 6, instructional time should focus on five critical areas: (1) connecting ratio and rate to whole number multiplication and division, and using concepts of ratio and rate to solve problems; (2) completing understanding of division of fractions and extending the notion of number to the system of rational numbers, which includes negative numbers; (3) writing, interpreting, and using expressions and equations; (4) developing understanding of statistical thinking; and (5) reasoning about geometric shapes and their measurements.

1. Students use reasoning about multiplication and division to solve ratio and rate problems about quantities. By viewing equivalent ratios and rates as deriving from, and extending, pairs of rows (or columns) in the multiplication table, and by analyzing simple drawings that indicate the relative size of quantities, students connect their understanding of multiplication and division with ratios and rates. Thus students expand the scope of problems for which they can use multiplication and division to solve problems, and they connect ratios and fractions. Students solve a wide variety of problems involving ratios and rates.
2. Students use the meaning of fractions, the meanings of multiplication and division, and the relationship between multiplication and division to understand and explain why the procedures for dividing fractions make sense. Students use these operations to solve problems. Students extend their previous understandings of number and the ordering of numbers to the full system of rational numbers, which includes negative rational numbers, and in particular negative integers. They reason about the order and absolute value of rational numbers and about the location of points in all four quadrants of the coordinate plane.
3. Students understand the use of variables in mathematical expressions. They write expressions and equations that correspond to given situations, evaluate expressions, and use expressions and formulas to solve problems. Students understand that expressions in different forms can be equivalent, and they use the properties of operations to rewrite expressions in equivalent forms. Students know that the solutions of an equation are the values of the variables that make the equation true. Students use properties of operations and the idea of maintaining the equality of both sides of an equation to solve simple one-step equations. Students construct and analyze tables, such as tables of quantities that are in equivalent ratios, and they use equations (such as $3x = y$) to describe relationships between quantities.
4. Building on and reinforcing their understanding of number, students begin to develop their ability to think statistically. Students recognize that a data distribution may not have a definite center and that different ways to measure center yield different values. The median measures center in the sense that it is roughly the middle value. The mean measures center in the sense that it is the value that each data point would take on if the total of the data values were redistributed equally, and also in the sense that it is a balance point. Students recognize that a measure of variability (interquartile range) can also be useful for summarizing data because two very different sets of data can have the same mean and median yet be distinguished by their variability. Students learn to describe and summarize numerical data sets, identifying clusters, peaks, gaps, and symmetry, considering the context in which the data were collected.
5. Students in grade 6 also build on their work with area in elementary school by reasoning about relationships among shapes to determine area, surface area, and volume. They find areas of right

triangles, other triangles, and special quadrilaterals by decomposing these shapes, rearranging or removing pieces, and relating the shapes to rectangles. Using these methods, students discuss, develop, and justify formulas for areas of triangles and parallelograms. Students find areas of polygons and surface areas of prisms and pyramids by decomposing them into pieces whose area they can determine. They reason about right rectangular prisms with fractional side lengths to extend formulas for the volume of a right rectangular prism to fractional side lengths. They prepare for work on scale drawings and constructions in grade 7 by drawing polygons in the coordinate plane.

The Standards for Mathematical Practice complement the content standards so that students increasingly engage with the subject matter as they grow in mathematical maturity and expertise throughout the elementary, middle, and high school years.

Grade 6 Overview

Ratios and Proportional Relationships

A. Understand ratio and rate concepts and use ratio reasoning to solve problems.

The Number System

A. Apply and extend previous understandings of multiplication and division to divide fractions by fractions.
B. Compute fluently with multi-digit numbers and find common factors and multiples.
C. Apply and extend previous understandings of numbers to the system of rational numbers.

Expressions and Equations

A. Apply and extend previous understandings of arithmetic to algebraic expressions.
B. Reason about and solve one-variable equations and inequalities.
C. Represent and analyze quantitative relationships between dependent and independent variables.

Geometry

A. Solve real-world and mathematical problems involving area, surface area, and volume.

Statistics and Probability

A. Develop understanding of statistical variability.
B. Summarize and describe distributions.

Standards for Mathematical Practice

1. Make sense of problems and persevere in solving them.
2. Reason abstractly and quantitatively.
3. Construct viable arguments and critique the reasoning of others.
4. Model with mathematics.
5. Use appropriate tools strategically.
6. Attend to precision.
7. Look for and make use of structure.
8. Look for and express regularity in repeated reasoning.

Grade 6 Content Standards

Ratios and Proportional Relationships 6.RP

A. Understand ratio and rate concepts and use ratio and rate reasoning to solve problems.

1. Understand the concept of a ratio including the distinctions between part:part and part:whole and the value of a ratio; part/part and part/whole. Use ratio language to describe a ratio relationship between two quantities.

   For example, The ratio of wings to beaks in the bird house at the zoo was 2:1, because for every two wings there was one beak; For every vote candidate A received, candidate C received nearly three votes, meaning that candidate C received three out of every four votes or ¾ of all votes.

2. Understand the concept of a unit rate a/b associated with a ratio $a:b$ with $b \neq 0$, and use rate language in the context of a ratio relationship, *including the use of units*.

   For example, This recipe has a ratio of three cups of flour to four cups of sugar, so there is ¾ cup of flour for each cup of sugar; We paid $75 for 15 hamburgers, which is a rate of five dollars per hamburger.[22]

3. Use ratio and rate reasoning to solve real-world and mathematical problems, e.g., by reasoning about tables of equivalent ratios, tape diagrams, double number line diagrams, or equations.
   - **a.** Make tables of equivalent ratios relating quantities with whole-number measurements. Find missing values in the tables, and plot the pairs of values on the coordinate plane. Use tables to compare ratios.
   - **b.** Solve unit rate problems, including those involving unit pricing, and constant speed.

   For example, if it took seven hours to mow four lawns, then, at that rate, how many lawns could be mowed in 35 hours? At what rate were lawns being mowed?

   - **c.** Find a percent of a quantity as a rate per 100 (e.g., 30% of a quantity means $^{30}/_{100}$ times the quantity); solve problems involving finding the whole, given a part and the percent.
   - **d.** Use ratio reasoning to convert measurement units within and between measurement systems; manipulate and transform units appropriately when multiplying or dividing quantities.

   For example, Malik is making a recipe, but he cannot find his measuring cups! He has, however, found a tablespoon. His cookbook says that 1 cup = 16 tablespoons. Explain how he could use the tablespoon to measure out the following ingredients: two cups of flour, ½ cup sunflower seed, and 1¼ cup of oatmeal.[23]

   - **e.** Solve problems that relate the mass of an object to its volume.

The Number System 6.NS

A. Apply and extend previous understandings of multiplication and division to divide fractions by fractions.

1. Interpret and compute quotients of fractions, and solve word problems involving division of fractions by fractions, e.g., by using visual fraction models and equations to represent the problem.

   For example, create a story context for ($^2/_3$) ÷ ($^3/_4$) and use a visual fraction model to show the quotient; use the relationship between multiplication and division to explain that ($^2/_3$) ÷ ($^3/_4$) = $^8/_9$ because $^3/_4$ of $^8/_9$ is $^2/_3$. In general, ($^a/_b$) ÷ ($^c/_d$) = $^{ad}/_{bc}$. How much chocolate will each person get if three people share $^1/_2$ lb. of chocolate equally? How many $^3/_4$-cup servings are in $^2/_3$ of a cup of yogurt? How wide is a rectangular strip of land with length $^3/_4$ mile and area $^1/_2$ square mile?

B. Compute fluently with multi-digit numbers and find common factors and multiples.

2. Fluently divide multi-digit numbers using the standard algorithm.
3. Fluently add, subtract, multiply, and divide multi-digit decimals using the standard algorithm for each operation.

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[22] Expectations for unit rates in this grade are limited to non-complex fractions.

[23] Example is from the Illustrative Mathematics Project: https://www.illustrativemathematics.org/content-standards/tasks/2174

4. Use prime factorization to find the greatest common factor of two whole numbers less than or equal to 100 and the least common multiple of two whole numbers less than or equal to 12. Use the distributive property to express a sum of two whole numbers 1–100 with a common factor as a multiple of a sum of two relatively prime numbers.

For example, express 36 + 8 as 4(9 + 2).

C. Apply and extend previous understandings of numbers to the system of rational numbers.

5. Understand that positive and negative numbers are used together to describe quantities having opposite directions or values (e.g., temperature above/below zero, elevation above/below sea level, credits/debits, and positive/negative electric charge). Use positive and negative numbers (whole numbers, fractions, and decimals) to represent quantities in real-world contexts, explaining the meaning of zero in each situation.
6. Understand a rational number as a point on the number line. Extend number line diagrams and coordinate axes familiar from previous grades to represent points on the line and in the plane with negative number coordinates.
    - **a.** Recognize opposite signs of numbers as indicating locations on opposite sides of 0 on the number line; recognize that the opposite of the opposite of a number is the number itself, e.g., –(–3) = 3, and that zero is its own opposite.
    - **b.** Understand signs of numbers in ordered pairs as indicating locations in quadrants of the coordinate plane; recognize that when two ordered pairs differ only by signs, the locations of the points are related by reflections across one or both axes.
    - **c.** Find and position integers and other rational numbers on a horizontal or vertical number line diagram; find and position pairs of integers and other rational numbers on a coordinate plane.
7. Understand ordering and absolute value of rational numbers.
    - **a.** Interpret statements of inequality as statements about the relative positions of two numbers on a number line diagram.

For example, interpret –3 > –7 as a statement that –3 is located to the right of –7 on a number line oriented from left to right.

- **b.** Write, interpret, and explain statements of order for rational numbers in real-world contexts.

For example, write –3°C > –7°C to express the fact that –3°C is warmer than –7°C.

- **c.** Understand the absolute value of a rational number as its distance from 0 on the number line; interpret absolute value as magnitude for a positive or negative quantity in a real-world situation.

For example, for an account balance of –30 dollars, write |–30| = 30 to describe the size of the debt in dollars.

- **d.** Distinguish comparisons of absolute value from statements about order.

For example, recognize that an account balance less than –30 dollars represents a debt greater than 30 dollars.

8. Solve real-world and mathematical problems by graphing points in all four quadrants of the coordinate plane. Include use of coordinates and absolute value to find distances between points with the same first coordinate or the same second coordinate.

Expressions and Equations 6.EE

A. Apply and extend previous understandings of arithmetic to algebraic expressions.

1. Write and evaluate numerical expressions involving whole-number exponents.
2. Write, read, and evaluate expressions in which letters stand for numbers.
    - **a.** Write expressions that record operations with numbers and with letters standing for numbers.

For example, express the calculation "Subtract y from 5" as 5 – y.

- **b.** Identify parts of an expression using mathematical terms (sum, term, product, factor, quotient, and coefficient); view one or more parts of an expression as a single entity.

For example, describe the expression 2(8 + 7) as a product of two factors; view (8 + 7) as both a single entity and a sum of two terms.

- **c.** Evaluate expressions at specific values of their variables. Include expressions that arise from formulas used in real-world problems. Perform arithmetic operations, including those involving whole-number exponents, in the conventional order when there are no parentheses to specify a particular order (Order of Operations).

For example, use the formulas $V = s^3$ and $A = 6s^2$ to find the volume and surface area of a cube with sides of length $s = ½$.

3. Apply the properties of operations to generate equivalent expressions.

For example, apply the distributive property to the expression 3(2 + x) to produce the equivalent expression 6 + 3x; apply the distributive property to the expression 24x + 18y to produce the equivalent expression 6(4x + 3y); apply properties of operations to y + y + y to produce the equivalent expression 3y.

4. Identify when two expressions are equivalent (i.e., when the two expressions name the same number regardless of which value is substituted into them).

For example, the expressions y + y + y and 3y are equivalent because they name the same number regardless of which number y stands for.

B. Reason about and solve one-variable equations and inequalities.

5. Understand solving an equation or inequality as a process of answering a question: Which values from a specified set, if any, make the equation or inequality true? Use substitution to determine whether a given number in a specified set makes an equation or inequality true.

6. Use variables to represent numbers and write expressions when solving a real-world or mathematical problem; understand that a variable can represent an unknown number, or, depending on the purpose at hand, any number in a specified set.

7. Solve real-world and mathematical problems by writing and solving equations of the form $x + p = q$ and $px = q$ for cases in which p, q, and x are all nonnegative rational numbers.

8. Write an inequality of the form $x > c$ or $x < c$ to represent a constraint or condition in a real-world or mathematical problem. Recognize that inequalities of the form $x > c$ or $x < c$ have infinitely many solutions; represent solutions of such inequalities on number line diagrams.

C. Represent and analyze quantitative relationships between dependent and independent variables.

9. Use variables to represent two quantities in a real-world problem that change in relationship to one another; write an equation to express one quantity, thought of as the dependent variable, in terms of the other quantity, thought of as the independent variable. Analyze the relationship between the dependent and independent variables using graphs and tables, and relate these to the equation.

For example, in a problem involving motion at constant speed, list and graph ordered pairs of distances and times, and write the equation d = 65t to represent the relationship between distance and time.

Geometry 6.G

A. Solve real-world and mathematical problems involving area, surface area, and volume.

1. Find the area of right triangles, other triangles, special quadrilaterals, and polygons by composing into rectangles or decomposing into triangles and other shapes; apply these techniques in the context of solving real-world and mathematical problems.

2. Find the volume of a right rectangular prism with fractional edge lengths by packing it with unit cubes of the appropriate unit fraction edge lengths, and show that the volume is the same as would be found by multiplying the edge lengths of the prism. Apply the formulas $V = lwh$ and $V = Bh$ to find volumes of right rectangular prisms with fractional edge lengths in the context of solving real-world and mathematical problems.

3. **3.** Draw polygons in the coordinate plane given coordinates for the vertices; use coordinates to find the length of a side joining points with the same first coordinate or the same second coordinate. Apply these techniques in the context of solving real-world and mathematical problems.
4. **4.** Represent three-dimensional figures using nets made up of rectangles and triangles, and use the nets to find the surface areas of these figures. Apply these techniques in the context of solving real-world and mathematical problems.

Statistics and Probability 6.SP

A. Develop understanding of statistical variability.

1. **1.** Recognize a statistical question as one that anticipates variability in the data related to the question and accounts for it in the answers.
   For example, "How old am I?" is not a statistical question, but "How old are the students in my school?" is a statistical question because one anticipates variability in students' ages.
2. **2.** Understand that a set of data collected to answer a statistical question has a distribution, which can be described by its center (median, mean, and/or mode), spread (range, interquartile range), and overall shape.
3. **3.** Recognize that a measure of center for a numerical data set summarizes all of its values with a single number, while a measure of variation describes how its values vary with a single number.

B. Summarize and describe distributions.

4. **4.** Display numerical data in plots on a number line, including dot plots, histograms, and box plots.
   - **a.** Read and interpret circle graphs.
5. **5.** Summarize numerical data sets in relation to their context, such as by:
   - **a.** Reporting the number of observations.
   - **b.** Describing the nature of the attribute under investigation, including how it was measured and its units of measurement.
   - **c.** Giving quantitative measures of center (median, and/or mean) and variability (range and/or interquartile range), as well as describing any overall pattern and any striking deviations from the overall pattern with reference to the context in which the data were gathered.
   - **d.** Relating the choice of measures of center and variability to the shape of the data distribution and the context in which the data were gathered.

Grade 7

Introduction

In grade 7, instructional time should focus on four critical areas: (1) developing understanding of and applying proportional relationships; (2) developing understanding of operations with rational numbers and working with expressions and linear equations; (3) solving problems involving scale drawings and informal geometric constructions, and working with two- and three-dimensional shapes to solve problems involving area, surface area, and volume; and (4) drawing inferences about populations based on samples.

1. Students extend their understanding of ratios and rates and develop understanding of proportionality to solve single- and multi-step problems. Students use their understanding of ratios, rates, and proportionality to solve a wide variety of percent problems, including those involving discounts, interest, taxes, tips, and percent increase or decrease. Students solve problems about scale drawings by relating corresponding lengths between the objects or by using the fact that relationships of lengths within an object are preserved in similar objects. Students graph proportional relationships and understand the unit rate informally as a measure of the steepness of the related line, called the slope. They distinguish proportional relationships from other relationships.
2. Students develop a unified understanding of number, recognizing fractions, decimals (that have a finite or a repeating decimal representation), and percents as different representations of rational numbers. Students extend addition, subtraction, multiplication, and division to all rational numbers, maintaining the properties of operations and the relationships between addition and subtraction, and multiplication and division. By applying these properties, and by viewing negative numbers in terms of everyday contexts (e.g., amounts owed or temperatures below zero), students explain and interpret the rules for adding, subtracting, multiplying, and dividing with negative numbers. They use the arithmetic of rational numbers as they formulate expressions and equations in one variable and use these equations to solve problems.
3. Students continue their work with area from grade 6, solving problems involving the area and circumference of a circle and surface area of three-dimensional objects. In preparation for work on congruence and similarity in grade 8 they reason about relationships among two-dimensional figures using scale drawings and informal geometric constructions, and they gain familiarity with the relationships between angles formed by intersecting lines. Students work with three-dimensional figures, relating them to two-dimensional figures by examining cross-sections. They solve real-world and mathematical problems involving area, surface area, and volume of two- and three-dimensional objects composed of triangles, quadrilaterals, polygons, cubes, and right prisms.
4. Students build on their previous work with single data distributions to compare two data distributions and address questions about differences between populations. They begin informal work with random sampling to generate data sets and learn about the importance of representative samples for drawing inferences.

The Standards for Mathematical Practice complement the content standards so that students increasingly engage with the subject matter as they grow in mathematical maturity and expertise throughout the elementary, middle, and high school years.

Grade 7 Overview

Ratios and Proportional Relationships

A. Analyze proportional relationships and use them to solve real-world and mathematical problems.

The Number System

A. Apply and extend previous understandings of operations with fractions to add, subtract, multiply, and divide rational numbers.

Expressions and Equations

A. Use properties of operations to generate equivalent expressions.

B. Solve real-life and mathematical problems using numerical and algebraic expressions and equations.

Geometry

A. Draw, construct and describe geometrical figures and describe the relationships between them.

B. Solve real-life and mathematical problems involving angle measure, area, surface area, and volume.

Standards for Mathematical Practice

1. Make sense of problems and persevere in solving them.
2. Reason abstractly and quantitatively.
3. Construct viable arguments and critique the reasoning of others.
4. Model with mathematics.
5. Use appropriate tools strategically.
6. Attend to precision.
7. Look for and make use of structure.
8. Look for and express regularity in repeated reasoning.

Statistics and Probability

A. Use random sampling to draw inferences about a population.

B. Draw informal comparative inferences about two populations.

C. Investigate chance processes and develop, use, and evaluate probability models.

Grade 7 Content Standards

Ratios and Proportional Relationships 7.RP

A. Analyze proportional relationships and use them to solve real-world and mathematical problems.

1. Compute unit rates associated with ratios of fractions, including ratios of lengths, areas, and other quantities measured in like or different units.

   For example, if a person walks ½ mile in each ¼ hour, compute the unit rate as the complex fraction ½/¼ miles per hour, equivalently 2 miles per hour.

2. Recognize and represent proportional relationships between quantities.
   - **a.** Decide whether two quantities are in a proportional relationship, e.g., by testing for equivalent ratios in a table, or graphing on a coordinate plane and observing whether the graph is a straight line through the origin.
   - **b.** Identify the constant of proportionality (unit rate) in tables, graphs, equations, diagrams, and verbal descriptions of proportional relationships.
   - **c.** Represent proportional relationships by equations.

   For example, if total cost t is proportional to the number n of items purchased at a constant price p, the relationship between the total cost and the number of items can be expressed as t = pn.

   - **d.** Explain what a point (x, y) on the graph of a proportional relationship means in terms of the situation, with special attention to the points (0, 0) and $(1, r)$ where r is the unit rate.

3. Use proportional relationships to solve multi-step ratio, rate, and percent problems.

   For example: simple interest, tax, price increases and discounts, gratuities and commissions, fees, percent increase and decrease, percent error.

The Number System 7.NS

A. Apply and extend previous understandings of operations with fractions to add, subtract, multiply, and divide rational numbers.

1. Apply and extend previous understandings of addition and subtraction to add and subtract integers and other rational numbers; represent addition and subtraction on a horizontal or vertical number line diagram.
   - **a.** Describe situations in which opposite quantities combine to make zero.

   For example, A hydrogen atom has zero charge because its two constituents are oppositely charged; If you open a new bank account with a deposit of $30 and then withdraw $30, you are left with a $0 balance.

   - **b.** Understand $p + q$ as the number located a distance $|q|$ from p, in the positive or negative direction depending on whether q is positive or negative. Show that a number and its opposite have a sum of 0 (are additive inverses). Interpret sums of rational numbers by describing real-world contexts.
   - **c.** Understand subtraction of rational numbers as adding the additive inverse, $p – q = p + (–q)$. Show that the distance between two rational numbers on the number line is the absolute value of their difference, and apply this principle in real-world contexts.
   - **d.** Apply properties of operations as strategies to add and subtract rational numbers.

2. Apply and extend previous understandings of multiplication and division and of fractions to multiply and divide integers and other rational numbers.
   - **a.** Understand that multiplication is extended from fractions to rational numbers by requiring that operations continue to satisfy the properties of operations, particularly the distributive property, leading to products such as $(–1)(–1) = 1$ and the rules for multiplying signed numbers. Interpret products of rational numbers by describing real-world contexts.

   - **b.** Understand that integers can be divided, provided that the divisor is not zero, and every quotient of integers (with non-zero divisor) is a rational number. If p and q are integers, then $-(^{p}/_{q}) = {}^{(-p)}/_{q} = {}^{p}/_{(-q)}$. Interpret quotients of rational numbers by describing real-world contexts.
   - **c.** Apply properties of operations as strategies to multiply and divide rational numbers.
   - **d.** Convert a rational number to a decimal using long division; know that the decimal form of a rational number terminates in 0s or eventually repeats.

3. Solve real-world and mathematical problems involving the four operations with integers and other rational numbers.[24]

Expressions and Equations 7.EE

A. Use properties of operations to generate equivalent expressions.

1. Apply properties of operations to add, subtract, factor, and expand linear expressions with rational coefficients.

For example, $4x + 2 = 2(2x + 1)$ and $-3(x - {}^{5}/_{3}) = -3x + 5$.

2. Understand that rewriting an expression in different forms in a problem context can shed light on the problem and how the quantities in it are related.

For example, $a + 0.05a = 1.05a$ means that "increase by 5%" is the same as "multiply by 1.05." A shirt at a clothing store is on sale for 20% off the regular price, "p". The discount can be expressed as 0.2p. The new price for the shirt can be expressed as p – 0.2p or 0.8p.

B. Solve real-life and mathematical problems using numerical and algebraic expressions and equations.

3. Solve multi-step real-life and mathematical problems posed with positive and negative rational numbers in any form (whole numbers, fractions, and decimals), using tools strategically. Apply properties of operations to calculate with numbers in any form; convert between forms as appropriate; and assess the reasonableness of answers using mental computation and estimation strategies.

For example, if a woman making $25 an hour gets a 10% raise, she will make an additional ${}^{1}/_{10}$ of her salary an hour, or $2.50, for a new salary of $27.50. If you want to place a towel bar 9¾ inches long in the center of a door that is 27½ inches wide, you will need to place the bar about 9 inches from each edge; This estimate can be used as a check on the exact computation.

4. Use variables to represent quantities in a real-world or mathematical problem, and construct simple equations and inequalities to solve problems by reasoning about the quantities.

   - **a.** Solve word problems leading to equations of the form $px + q = r$ and $p(x \div q) = r$, where p, q, and r are specific rational numbers. Solve equations of these forms fluently. Compare an algebraic solution to an arithmetic solution, identifying the sequence of the operations used in each approach.

For example, the perimeter of a rectangle is 54 cm. Its length is 6 cm. What is its width?

   - **b.** Solve word problems leading to inequalities of the form $px + q > r$ or $px + q < r$, where p, q, and r are specific rational numbers. Graph the solution set of the inequality and interpret it in the context of the problem.

For example, as a salesperson, you are paid $50 per week plus $3 per sale. This week you want your pay to be at least $100. Write an inequality for the number of sales you need to make, and describe the solutions.

   - **c.** Extend analysis of patterns to include analyzing, extending, and determining an expression for simple arithmetic and geometric sequences (e.g., compounding, increasing area), using tables, graphs, words, and expressions.

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[24] Computations with rational numbers extend the rules for manipulating fractions to complex fractions.

Geometry 7.G

A. Draw, construct, and describe geometrical figures and describe the relationships between them.

1. Solve problems involving scale drawings of geometric figures, such as computing actual lengths and areas from a scale drawing and reproducing a scale drawing at a different scale.

2. Draw (freehand, with ruler and protractor, and with technology) two-dimensional geometric shapes with given conditions. Focus on constructing triangles from three measures of angles or sides, noticing when the conditions determine a unique triangle, more than one triangle, or no triangle.

3. Describe the shape of the two-dimensional face of the figure that results from slicing three-dimensional figures, as in plane sections of right rectangular prisms and right rectangular pyramids.

B. Solve real-life and mathematical problems involving angle measure, area, surface area, and volume.

4. Circles and measurement:

- **a.** Know that a circle is a two-dimensional shape created by connecting all of the points equidistant from a fixed point called the center of the circle.
- **b.** Understand and describe the relationships among the radius, diameter, and circumference of a circle.
- **c.** Understand and describe the relationship among the radius, diameter, and area of a circle.
- **d.** Know the formulas for the area and circumference of a circle and use them to solve problems.
- **e.** Give an informal derivation of the relationship between the circumference and area of a circle.

5. Use facts about supplementary, complementary, vertical, and adjacent angles in a multi-step problem to write simple equations and use them to solve for an unknown angle in a figure.

6. Solve real-world and mathematical problems involving area, volume, and surface area of two- and three-dimensional objects composed of triangles, quadrilaterals, polygons, cubes, and right prisms.

Statistics and Probability 7.SP

A. Use random sampling to draw inferences about a population.

1. Understand that statistics can be used to gain information about a population by examining a sample of the population; Generalizations about a population from a sample are valid only if the sample is representative of that population. Understand that random sampling tends to produce representative samples and support valid inferences.

2. Use data from a random sample to draw inferences about a population with an unknown characteristic of interest. Generate multiple samples (or simulated samples) of the same size to gauge the variation in estimates or predictions.

For example, estimate the mean word length in a book by randomly sampling words from the book; predict the winner of a school election based on randomly sampled survey data. Gauge how far off the estimate or prediction might be.

B. Draw informal comparative inferences about two populations.

3. Informally assess the degree of visual overlap of two numerical data distributions with similar variabilities, measuring the difference between the centers by expressing it as a multiple of a measure of variability.

For example, the mean height of players on the basketball team is 10 cm greater than the mean height of players on the soccer team and both distributions have similar variability (mean absolute deviation) of about 5 cm. The difference between the mean heights of the two teams (10 cm) is about twice the variability (5 cm) on either team. On a dot plot, the separation between the two distributions of heights is noticeable.

4. Use measures of center and measures of variability for numerical data from random samples to draw informal comparative inferences about two populations.

For example, decide whether the words in a chapter of a seventh-grade science book are generally longer than the words in a chapter of a fourth-grade science book.

C. Investigate chance processes and develop, use, and evaluate probability models.

5. Understand that the probability of a chance event is a number between 0 and 1 that expresses the likelihood of the event occurring. Larger numbers indicate greater likelihood. A probability near 0 indicates an unlikely event, a probability around ½ indicates an event that is neither unlikely nor likely, and a probability near 1 indicates a likely event.
6. Approximate the probability of a chance event by collecting data on the chance process that produces it and observing its long-run relative frequency, and predict the approximate relative frequency given the probability.

   For example, when rolling a number cube 600 times, predict that a 3 or 6 would be rolled roughly 200 times, but probably not exactly 200 times.
7. Develop a probability model and use it to find probabilities of events. Compare probabilities from a model to observed frequencies; if the agreement is not good, explain possible sources of the discrepancy.
   - **a.** Develop a uniform probability model by assigning equal probability to all outcomes, and use the model to determine probabilities of events.

   For example, if a student is selected at random from a class, find the probability that Jane will be selected and the probability that a girl will be selected.
   - **b.** Develop a probability model (which may not be uniform) by observing frequencies in data generated from a chance process.

   For example, find the approximate probability that a spinning penny will land heads up or that a tossed paper cup will land open-end down. Do the outcomes for the spinning penny appear to be equally likely based on the observed frequencies?
8. Find probabilities of compound events using organized lists, tables, tree diagrams, and simulation.
   - **a.** Understand that, just as with simple events, the probability of a compound event is the fraction of outcomes in the sample space for which the compound event occurs.
   - **b.** Represent sample spaces for compound events using methods such as organized lists, tables, and tree diagrams. For an event described in everyday language (e.g., “rolling double sixes”), identify the outcomes in the sample space which compose the event.
   - **c.** Design and use a simulation to generate frequencies for compound events.

   For example, use random digits as a simulation tool to approximate the answer to the question: If 40% of donors have type A blood, what is the probability that it will take at least four donors to find one with type A blood?

Grade 8

Introduction

In grade 8, instructional time should focus on three critical areas: (1) formulating and reasoning about expressions and equations, including modeling an association in bivariate data with a linear equation and solving linear equations and systems of linear equations; (2) grasping the concept of a function and using functions to describe quantitative relationships; and (3) analyzing two- and three-dimensional space and figures using distance, angle, similarity, and congruence, and understanding and applying the Pythagorean Theorem.

1. Students use linear equations and systems of linear equations to represent, analyze, and solve a variety of problems. Students recognize equations for proportions ($y/x = m$ or $y = mx$) as special linear equations ($y = mx + b$), understanding that the constant of proportionality (m) is the slope, and the graphs are lines through the origin. They understand that the slope (m) of a line is a constant rate of change, so that if the input or x-coordinate changes by an amount A, the output or y-coordinate changes by the amount m x A. Students also use a linear equation to describe the association between two quantities in bivariate data (such as arm span vs. height for students in a classroom). At this grade, fitting the model and assessing its fit to the data are done informally. Interpreting the model in the context of the data requires students to express a relationship between the two quantities in question and to interpret components of the relationship (such as slope and y-intercept) in terms of the situation.

   Students strategically choose and efficiently implement procedures to solve linear equations in one variable, understanding that when they use the properties of equality and the concept of logical equivalence, they maintain the solutions of the original equation. Students solve systems of two linear equations in two variables and relate the systems to pairs of lines in the plane; these intersect, are parallel, or are the same line. Students use linear equations, systems of linear equations, linear functions, and their understanding of slope of a line to analyze situations and solve problems.

2. Students grasp the concept of a function as a rule that assigns to each input exactly one output. They understand that functions describe situations where one quantity determines another. They can translate among representations and partial representations of functions (noting that tabular and graphical representations may be partial representations), and they describe how aspects of the function are reflected in the different representations.
3. Students use ideas about distance and angles, how they behave under translations, rotations, reflections, and dilations, and ideas about congruence and similarity to describe and analyze two-dimensional figures and to solve problems. Students show that the sum of the angles in a triangle is the angle formed by a straight line, and that various configurations of lines give rise to similar triangles because of the angles created when a transversal cuts parallel lines. Students understand the statement of the Pythagorean Theorem and its converse, and can explain why the Pythagorean Theorem holds, for example, by decomposing a square in two different ways. They apply the Pythagorean Theorem to find distances between points on the coordinate plane, to find lengths, and to analyze polygons. Students complete their work on volume by solving problems involving cones, cylinders, and spheres.

The Standards for Mathematical Practice complement the content standards so that students increasingly engage with the subject matter as they grow in mathematical maturity and expertise throughout the elementary, middle, and high school years.

Grade 8 Overview

The Number System

A. Know that there are numbers that are not rational, and approximate them by rational numbers.

Expressions and Equations

A. Work with radicals and integer exponents.
B. Understand the connections between proportional relationships, lines, and linear equations.
C. Analyze and solve linear equations and pairs of simultaneous linear equations.

Functions

A. Define, evaluate, and compare functions.
B. Use functions to model relationships between quantities.

Geometry

A. Understand congruence and similarity using physical models, transparencies, or geometry software.
B. Understand and apply the Pythagorean Theorem.
C. Solve real-world and mathematical problems involving volume of cylinders, cones, and spheres.

Statistics and Probability

A. Investigate patterns of association in bivariate data.

Standards for Mathematical Practice

1. Make sense of problems and persevere in solving them.
2. Reason abstractly and quantitatively.
3. Construct viable arguments and critique the reasoning of others.
4. Model with mathematics.
5. Use appropriate tools strategically.
6. Attend to precision.
7. Look for and make use of structure.
8. Look for and express regularity in repeated reasoning.

Grade 8 Content Standards

The Number System 8.NS

A. Know that there are numbers that are not rational, and approximate them by rational numbers.

1. Know that numbers that are not rational are called irrational. Understand informally that every number has a decimal expansion. For rational numbers show that the decimal expansion repeats eventually, and convert a decimal expansion which repeats eventually into a rational number.
2. Use rational approximations of irrational numbers to compare the size of irrational numbers, locate them approximately on a number line diagram, and estimate the value of expressions (e.g., π^2).

   For example, by truncating the decimal expansion of $\sqrt{2}$ ***show that*** $\sqrt{2}$ ***is between 1 and 2, then between 1.4 and 1.5, and explain how to continue on to get better approximations.***

Expressions and Equations 8.EE

A. Work with radicals and integer exponents.

1. Know and apply the properties of integer exponents to generate equivalent numerical expressions.

   For example, $3^2 \times 3^{-5} = 3^{-3} = 1/3^3 = 1/27$.
2. Use square root and cube root symbols to represent solutions to equations of the form $x^2 = p$ and $x^3 = p$, where p is a positive rational number. Evaluate square roots of small perfect squares and cube roots of small perfect cubes. Know that $\sqrt{2}$ is irrational.
3. Use numbers expressed in the form of a single digit multiplied by an integer power of 10 to estimate very large or very small quantities, and express how many times as much one is than the other.

   For example, estimate the population of the United States as 3×10^8 ***and the population of the world as*** 7×10^9***, and determine that the world population is more than 20 times larger.***
4. Perform operations with numbers expressed in scientific notation, including problems where both decimal and scientific notation are used. Use scientific notation and choose units of appropriate size for measurements of very large or very small quantities (e.g., use millimeters per year for seafloor spreading). Interpret scientific notation that has been generated by technology.

B. Understand the connections between proportional relationships, lines, and linear equations.

5. Graph proportional relationships, interpreting the unit rate as the slope of the graph. Compare two different proportional relationships represented in different ways.

   For example, compare a distance-time graph to a distance-time equation to determine which of two moving objects has greater speed.
6. Use similar triangles to explain why the slope m is the same between any two distinct points on a non-vertical line in the coordinate plane. Derive the equation $y = mx$ for a line through the origin and the equation $y = mx + b$ for a line intercepting the vertical axis at b.

C. Analyze and solve linear equations and pairs of simultaneous linear equations.

7. Solve linear equations in one variable.
   - **a.** Give examples of linear equations in one variable with one solution, infinitely many solutions, or no solutions. Show which of these possibilities is the case by successively transforming the given equation into simpler forms, until an equivalent equation of the form $x = a$, $a = a$, or $a = b$ results (where a and b are different numbers).
   - **b.** Solve linear equations with rational number coefficients, including equations whose solutions require expanding expressions using the distributive property and collecting like terms.
8. Analyze and solve pairs of simultaneous linear equations.
   - **a.** Understand that solutions to a system of two linear equations in two variables correspond to points of intersection of their graphs, because points of intersection satisfy both equations simultaneously.
   - **b.** Solve systems of two linear equations in two variables algebraically (using substitution and elimination strategies), and estimate solutions by graphing the equations. Solve simple cases by inspection.

   For example, $3x + 2y = 5$ ***and*** $3x + 2y = 6$ ***have no solution because*** $3x + 2y$ ***cannot simultaneously be 5 and 6.***

  - **c.** Solve real-world and mathematical problems leading to two linear equations in two variables.

    For example, given coordinates for two pairs of points, determine whether the line through the first pair of points intersects the line through the second pair.

Functions 8.F

A. Define, evaluate, and compare functions.

1. Understand that a function is a rule that assigns to each input exactly one output. The graph of a function is the set of ordered pairs consisting of an input and the corresponding output.[25]
2. Compare properties of two functions each represented in a different way (algebraically, graphically, numerically in tables, or by verbal descriptions).

   For example, given a linear function represented by a table of values and a linear function represented by an algebraic expression, determine which function has the greater rate of change.
3. Interpret the equation $y = mx + b$ as defining a linear function whose graph is a straight line; give examples of functions that are not linear.

   For example, the function $A = s^2$ giving the area of a square as a function of its side length is not linear because its graph contains the points (1, 1), (2, 4) and (3, 9), which are not on a straight line.

B. Use functions to model relationships between quantities.

4. Construct a function to model a linear relationship between two quantities. Determine the rate of change and initial value of the function from a description of a relationship or from two (x, y) values, including reading these from a table or from a graph. Interpret the rate of change and initial value of a linear function in terms of the situation it models, and in terms of its graph or a table of values.
5. Describe qualitatively the functional relationship between two quantities by analyzing a graph (e.g., where the function is increasing or decreasing, linear or nonlinear). Sketch a graph that exhibits the qualitative features of a function that has been described verbally.

Geometry 8.G

A. Understand congruence and similarity using physical models, transparencies, or geometry software.

1. Verify experimentally the properties of rotations, reflections, and translations:
   - **a.** Lines are transformed to lines, and line segments to line segments of the same length.
   - **b.** Angles are transformed to angles of the same measure.
   - **c.** Parallel lines are transformed to parallel lines.
2. Understand that a two-dimensional figure is congruent to another if the second can be obtained from the first by a sequence of rotations, reflections, and translations. Given two congruent figures, describe a sequence that exhibits the congruence between them.
3. Describe the effects of dilations, translations, rotations, and reflections on two-dimensional figures using coordinates.
4. Understand that a two-dimensional figure is similar to another if the second can be obtained from the first by a sequence of rotations, reflections, translations, and dilations. Given two similar two-dimensional figures, describe a sequence that exhibits the similarity between them.
5. Use informal arguments to establish facts about the angle sum and exterior angle of triangles, about the angles created when parallel lines are cut by a transversal, and the angle-angle criterion for similarity of triangles.

   For example, arrange three copies of the same triangle so that the sum of the three angles appears to form a line, and give an argument in terms of transversals why this is so.

B. Understand and apply the Pythagorean Theorem.

6. **a.** Understand the relationship among the sides of a right triangle.
   b. Analyze and justify the Pythagorean Theorem and its converse using pictures, diagrams, narratives, or models.
7. Apply the Pythagorean Theorem to determine unknown side lengths in right triangles in real-world and mathematical problems in two and three dimensions.
8. Apply the Pythagorean Theorem to find the distance between two points in a coordinate system.

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[25] Function notation is not required in grade 8.

C. Solve real-world and mathematical problems involving volume of cylinders, cones, and spheres.

9. Know the formulas for the volumes of cones, cylinders, and spheres, and use them to solve real-world and mathematical problems.

Statistics and Probability 8.SP

A. Investigate patterns of association in bivariate data.

1. Construct and interpret scatter plots for bivariate measurement data to investigate patterns of association between two quantities. Describe patterns such as clustering, outliers, positive or negative association, linear association, and nonlinear association.

2. Know that straight lines are widely used to model relationships between two quantitative variables. For scatter plots that suggest a linear association, informally fit a straight line and informally assess the model fit by judging the closeness of the data points to the line.

3. Use the equation of a linear model to solve problems in the context of bivariate measurement data, interpreting the slope and intercept.

For example, in a linear model for a biology experiment, interpret a slope of 1.5 cm/hr as meaning that an additional hour of sunlight each day is associated with an additional 1.5 cm in mature plant height.

4. Understand that patterns of association can also be seen in bivariate categorical data by displaying frequencies and relative frequencies in a two-way table. Construct and interpret a two-way table summarizing data on two categorical variables collected from the same subjects. Use relative frequencies calculated for rows or columns to describe possible association between the two variables.

For example, collect data from students in your class on whether or not they have a curfew on school nights and whether or not they have assigned chores at home. Is there evidence that those who have a curfew also tend to have chores?

The High School Standards for Mathematical Content

Introduction

The organization of the high school sections of this Framework is outlined below, followed by a detailed description of each section.

I. High School Content Standards organized by:

- **A. Conceptual Categories**
- **B. Model High School Courses**
  - ***1. Traditional Pathway***
    - **Algebra I**
    - **Geometry**
    - **Algebra II**
  - ***2. Integrated Pathway***
    - **Mathematics I**
    - **Mathematics II**
    - **Mathematics III**
  - ***3. Advanced Courses***
    - **Model Precalculus**
    - **Model Advanced Quantitative Reasoning**

II. Guidance for Making Decisions about Course Sequences and the Model Algebra I Course

I. The High School Content Standards

The high school content standards are organized in two ways: (1) by conceptual category and (2) by model courses. The high school content standards specify the mathematics that all students should study in order to be prepared for college, career, and civic engagement. All standards without a (+) symbol should be in the common mathematics curriculum for all college and career ready students.

The high school standards indicated by a (+) symbol contain additional mathematical content that high school students should learn in order to be prepared to take advanced courses such as calculus, advanced statistics, or discrete mathematics. Standards with a (+) symbol may also appear in courses intended for all students but, in these cases, are optional and indicate a standard that is beyond college and career readiness.

1. High School Content Standards Organized by Conceptual Category

The content of the high school standards presented by conceptual categories portrays a coherent view of learning progressions that develop students' mathematical knowledge, skills, and understanding through the high school years. For example: a student's work with the concept of functions crosses a number of traditional course boundaries, potentially up through and including calculus. Similar to the grade level content standards, each conceptual category (except Modeling) is further subdivided into several domains, and each domain is subdivided into clusters.

The conceptual categories are:

- **Number and Quantity (N)**
- **Algebra (A)**
- **Functions (F)**
- **Modeling (★)**
- **Geometry (G)**
- **Statistics and Probability (S)**

2. High School Content Standards Organized by Model Courses

The grades 9–12 high school mathematics standards presented by conceptual categories provide guidance on what students are expected to learn in order to be prepared for college and careers. These standards have been configured into eight high school courses. These model high school courses, organized into a Traditional Pathway and Integrated Pathway and advanced course work, represent a smooth transition from the PreK–8 grade standards. All of the content of the college and career ready high school content standards presented by Conceptual Categories is included in the appropriate locations within the three model courses of both Model Pathways. In this Framework, the wording of a Conceptual Category standard may be different in the model courses in which it is covered in order to clarify the content expectations for that particular model course. Note: In the 2010 Framework, footnotes were included in order to clarify content expectations; however, this Framework incorporates the clarifying language directly into the wording of the model course standards for ease of understanding.

The grade 8 standards are rigorous; students are expected to learn about linear relationships and equations to begin the study of functions and comparing rational and irrational numbers. In addition, the statistics presented in the grade 8 standards are sophisticated and include connecting linear relations with the representation of bivariate data. The Model Algebra I and Model Mathematics I courses progress from these concepts and skills, and focus on quadratic and exponential functions. Some students may master the grade 8 standards earlier than eighth grade, which would enable these students to take the high school Model Algebra I course or Model Mathematics I course in eighth grade. Students completing either Model Pathway are prepared for additional courses, such as the model-advanced courses that follow the three courses in either model pathway. Model advanced courses are comprised of the higher-level mathematics standards (+) in the conceptual categories.

Model Pathways and the Model High School Courses presented in this Framework:

- **Model Traditional Pathway**
    - **Model Algebra I (AI)**
    - **Model Geometry (GEO)**
    - **Model Algebra II (AII)**
- **Model Integrated Pathway**
    - **Model Mathematics I (MI)**
    - **Model Mathematics II (MII)**
    - **Model Mathematics III (MIII)**
- **Advanced Model Courses**
    - **Model Precalculus (PC)**
    - **Model Advanced Quantitative Reasoning (AQR)**

The Model Traditional Pathway reflects the approach typically seen in the U.S., consisting of two Model Algebra courses (I and II) with some Statistics and Probability standards included, and a Model Geometry course, with some Number and Quantity standards and some Statistics and Probability standards included. The Model Integrated Pathway reflects the approach typically seen internationally, consisting of a sequence of three model courses, each of which includes Number and Quantity, Algebra, Functions, Geometry, and Statistics and Probability standards. Appendix III provides a table for each conceptual category. Each table shows the distribution of the content standards across the eight Model Courses. Districts and schools maintain the flexibility to distribute standards across courses in other ways; the model courses are not the only possible designs.

II. Guidance for Making Decisions about Course Sequences and the Model Algebra I Course

Following the Model Course section, this Framework presents information and resources to ground discussions and decision-making about course-taking sequences in middle and high school and presents multiple options for students ready to move at an accelerated rate, including taking the Model High School Algebra I course in grade 8.

Connecting Content Standards and the Standards for Mathematical Practice

While the Model Pathways and Model Courses organize the Content Standards into possible model pathways to college, career, and civic readiness, the Content Standards must be connected to the Standards for Mathematical Practice to ensure that students increasingly engage with the subject matter as they grow in mathematical maturity and expertise. (Appendix II provides descriptions of the Standards for Mathematical Practice for three grade-spans: Pre-K–5, 6–8, and 9–12.)

High School Content Standards by Conceptual Categories

Content Standards by Conceptual Category Identifiers/Coding

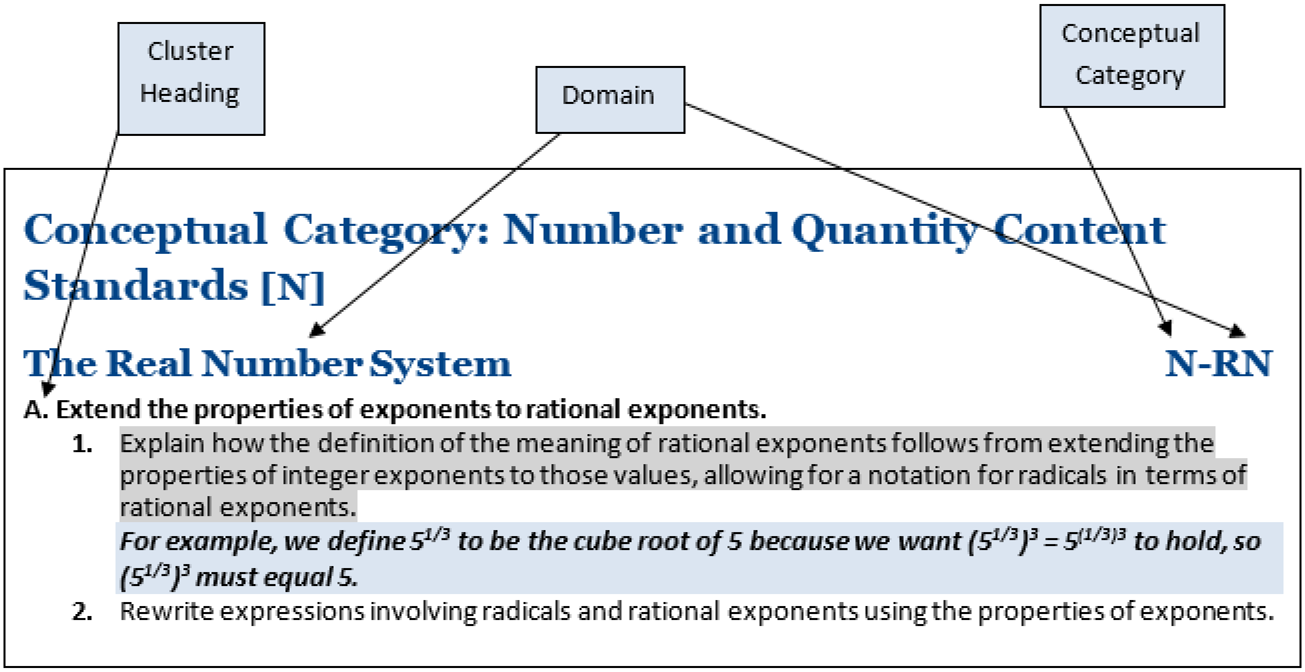
The content standards presented by conceptual categories are built on mathematical learning progressions informed by research on cognitive development and by the logical structure of mathematics. These progressions provide the foundation for the grades 9–12 high school content standards. In this section, the standards are organized by Conceptual Categories.

The Conceptual Categories are:

- **Number and Quantity (N)**
- **Algebra (A)**
- **Functions (F)**
- **Modeling (★)**
- **Geometry (G)**
- **Statistics and Probability (S)**

The content standards within the Conceptual Categories were then configured into eight high school courses.

The code for each high school conceptual category standard begins with the identifier for the conceptual category code (N, A, F, G, S), followed by the domain code, and the standard number, as shown below.



Conceptual Category: Number and Quantity Content Standards [N]

The Real Number System N-RN

A. Extend the properties of exponents to rational exponents.

1. Explain how the definition of the meaning of rational exponents follows from extending the properties of integer exponents to those values, allowing for a notation for radicals in terms of rational exponents.
   For example, we define $5^{1/3}$ to be the cube root of 5 because we want $(5^{1/3})^3 = 5^{(1/3)3}$ to hold, so $(5^{1/3})^3$ must equal 5.
2. Rewrite expressions involving radicals and rational exponents using the properties of exponents.

The standard highlighted above is identified as N-RN.A.1, identifying it as a standard in the Number and Quantity conceptual category (N-) within that category's Real Number System domain (RN), and as the first standard in that domain and in that cluster (A.1). All of the standards in this Framework use a common coding system.

Conceptual Category: Number and Quantity [N]

Introduction

Numbers and Number Systems

During the years from kindergarten to eighth grade, students must repeatedly extend their conception of number. At first, "number" means "counting number": 1, 2, 3.... Soon after that, 0 is used to represent "none" and the whole numbers are formed by the counting numbers together with zero. The next extension is fractions. At first, fractions are barely numbers and tied strongly to pictorial representations. Yet by the time students understand division of fractions, they have a strong concept of fractions as numbers and have connected them, via their decimal representations, with the Base Ten system used to represent the whole numbers. During middle school, fractions are augmented by negative fractions to form the rational numbers. In grade 8, students extend this system once more, augmenting the rational numbers with the irrational numbers to form the real numbers. In high school, students will be exposed to yet another extension of number, when the real numbers are augmented by the imaginary numbers to form the complex numbers. (See Illustration 1 in the Glossary.)

With each extension of number, the meanings of addition, subtraction, multiplication, and division are extended. In each new number system—integers, rational numbers, real numbers, and complex numbers—the four operations stay the same in two important ways: They have the Commutative, Associative, and Distributive properties and their new meanings are consistent with their previous meanings.

Extending the properties of whole-number exponents leads to new and productive notation. For example, properties of whole-number exponents suggest that $(5^{1/3})^3$ should be $5^{(1/3)3} = 5^1 = 5$ and that $5^{1/3}$ should be the cube root of 5.

Calculators, spreadsheets, and computer algebra systems can provide ways for students to become better acquainted with these new number systems and their notation. They can be used to generate data for numerical experiments, to help understand the workings of matrix, vector, and complex number algebra, and to experiment with non-integer exponents.

Quantities

In real-world problems, the answers are usually not numbers but quantities: numbers with units, which involve measurement. In their work in measurement up through grade 8, students primarily measure commonly used attributes such as length, area, and volume. In high school, students encounter a wider variety of units in modeling, e.g., acceleration, currency conversions, derived quantities such as person-hours and heating degree-days, social science rates such as per-capita income, and rates in everyday life such as points scored per game or batting averages. They also encounter novel situations in which they themselves must conceive the attributes of interest. For example, to find a good measure of overall highway safety, they might propose measures such as fatalities per year, fatalities per year per driver, or fatalities per vehicle-mile traveled. Such a conceptual process is sometimes called quantification. Quantification is important for science, as when surface area suddenly "stands out" as an important variable in evaporation. Quantification is also important for companies, which must conceptualize relevant attributes and create or choose suitable measures for them.

Conceptual Category: Number and Quantity Overview [N]

The Real Number System

- **A.** Extend the properties of exponents to rational exponents.
- **B.** Use properties of rational and irrational numbers.

Quantities

- **A.** Reason quantitatively and use units to solve problems.

The Complex Number System

- **A.** Perform arithmetic operations with complex numbers.
- **B.** Represent complex numbers and their operations on the complex plane.
- **C.** Use complex numbers in polynomial identities and equations.

Vector and Matrix Quantities

- **A.** Represent and model with vector quantities.
- **B.** Perform operations on vectors.
- **C.** Perform operations on matrices and use matrices in applications.

Standards for Mathematical Practice

1. Make sense of problems and persevere in solving them.
2. Reason abstractly and quantitatively.
3. Construct viable arguments and critique the reasoning of others.
4. Model with mathematics.
5. Use appropriate tools strategically.
6. Attend to precision.
7. Look for and make use of structure.
8. Look for and express regularity in repeated reasoning.

Conceptual Category: Number and Quantity Content Standards [N]

The Real Number System N-RN

A. Extend the properties of exponents to rational exponents.

1. Explain how the definition of the meaning of rational exponents follows from extending the properties of integer exponents to those values, allowing for a notation for radicals in terms of rational exponents. ***For example, we define $5^{1/3}$ to be the cube root of 5 because we want $(5^{1/3})3 = 5^{(1/3)3}$ to hold, so $(5^{1/3})^3$ must equal 5.***
2. Rewrite expressions involving radicals and rational exponents using the properties of exponents.

B. Use properties of rational and irrational numbers.

3. Explain why the sum or product of two rational numbers is rational; that the sum of a rational number and an irrational number is irrational; and that the product of a nonzero rational number and an irrational number is irrational.

Quantities N-Q

A. Reason quantitatively and use units to solve problems.

1. Use units as a way to understand problems and to guide the solution of multi-step problems; choose and interpret units consistently in formulas; choose and interpret the scale and the origin in graphs and data displays.★
2. Define appropriate quantities for the purpose of descriptive modeling.★
3. Choose a level of accuracy appropriate to limitations on measurement when reporting quantities.★
    - **a.** Describe the effects of approximate error in measurement and rounding on measurements and on computed values from measurements. Identify significant figures in recorded measures and computed values based on the context given and the precision of the tools used to measure.★

The Complex Number System N-CN

A. Perform arithmetic operations with complex numbers.

1. Know there is a complex number i such that $i^2 = -1$, and every complex number has the form $a + bi$ with a and b real.
2. Use the relation $i^2 = -1$ and the Commutative, Associative, and Distributive properties to add, subtract, and multiply complex numbers.
3. (+) Find the conjugate of a complex number; use conjugates to find moduli and quotients of complex numbers.

B. Represent complex numbers and their operations on the complex plane.

4. (+) Represent complex numbers on the complex plane in rectangular and polar form (including real and imaginary numbers), and explain why the rectangular and polar forms of a given complex number represent the same number.
5. (+) Represent addition, subtraction, multiplication, and conjugation of complex numbers geometrically on the complex plane; use properties of this representation for computation. ***For example, $(-1+\sqrt{3i})^3=8$ because $(-1+\sqrt{3i})$ has modulus 2 and argument 120°.***
6. (+) Calculate the distance between numbers in the complex plane as the modulus of the difference, and the midpoint of a segment as the average of the numbers at its endpoints.

C. Use complex numbers in polynomial identities and equations.

7. Solve quadratic equations with real coefficients that have complex solutions.
8. (+) Extend polynomial identities to the complex numbers. ***For example, rewrite $x^2 + 4$ as $(x + 2i)(x – 2i)$.***
9. (+) Know the Fundamental Theorem of Algebra; show that it is true for quadratic polynomials.

Vector and Matrix Quantities

N-VM

A. Represent and model with vector quantities.

1. (+) Recognize vector quantities as having both magnitude and direction. Represent vector quantities by directed line segments, and use appropriate symbols for vectors and their magnitudes (e.g., $\mathbf{v}$, $|\mathbf{v}|$, $||\,\mathbf{v}\,||$, v).
2. (+) Find the components of a vector by subtracting the coordinates of an initial point from the coordinates of a terminal point.
3. (+) Solve problems involving velocity and other quantities that can be represented by vectors.

B. Perform operations on vectors.

4. (+) Add and subtract vectors.
    - **a.** (+) Add vectors end-to-end, component-wise, and by the parallelogram rule. Understand that (+) the magnitude of a sum of two vectors is typically not the sum of the magnitudes.
    - **b.** (+) Given two vectors in magnitude and direction form, determine the magnitude and direction of their sum.
    - **c.** (+) Understand vector subtraction $\mathbf{v} - \mathbf{w}$ as $\mathbf{v} + (-\mathbf{w})$, where $-\mathbf{w}$ is the additive inverse of $\mathbf{w}$, with the same magnitude as $\mathbf{w}$ and pointing in the opposite direction. Represent vector subtraction graphically by connecting the tips in the appropriate order, and perform vector subtraction component-wise.
5. (+) Multiply a vector by a scalar.
    - **a.** (+) Represent scalar multiplication graphically by scaling vectors and possibly reversing their direction; perform scalar multiplication component-wise, e.g., as $c(v_x, v_y) = (cv_x, cv_y)$.
    - **b.** (+) Compute the magnitude of a scalar multiple $c\mathbf{v}$ using $||\,c\mathbf{v}|| = |c|\mathbf{v}$. Compute the direction of $c\mathbf{v}$ knowing that when $|c|\mathbf{v} \neq 0$, the direction of $c\mathbf{v}$ is either along $\mathbf{v}$ (for $c > 0$) or against $\mathbf{v}$ (for $c < 0$).

C. Perform operations on matrices and use matrices in applications.

6. (+) Use matrices to represent and manipulate data, e.g., to represent payoffs or incidence relationships in a network.
7. (+) Multiply matrices by scalars to produce new matrices, e.g., as when all of the payoffs in a game are doubled.
8. (+) Add, subtract, and multiply matrices of appropriate dimensions.
9. (+) Understand that, unlike multiplication of numbers, matrix multiplication for square matrices is not a Commutative operation, but still satisfies the Associative and Distributive properties.
10. (+) Understand that the zero and identity matrices play a role in matrix addition and multiplication similar to the role of 0 and 1 in the real numbers. The determinant of a square matrix is nonzero if and only if the matrix has a multiplicative inverse.
11. (+) Multiply a vector (regarded as a matrix with one column) by a matrix of suitable dimensions to produce another vector. Work with matrices as transformations of vectors.
12. (+) Work with 2×2 matrices as transformations of the plane, and interpret the absolute value of the determinant in terms of area.

Conceptual Category: Algebra [A]

Introduction

Expressions

An expression is a record of a computation with numbers, symbols that represent numbers, arithmetic operations, exponentiation, and, at more advanced levels, the operation of evaluating a function. Conventions about the use of parentheses and the Order of Operations assure that each expression is unambiguous. Creating an expression that describes a computation involving a general quantity requires the ability to express the computation in general terms, abstracting from specific instances.

Reading an expression with comprehension involves analysis of its underlying structure. This may suggest a different but equivalent way of writing the expression that exhibits some different aspect of its meaning. For example, $p + 0.05p$ can be interpreted as the addition of a 5% tax to a price p. Rewriting $p + 0.05p$ as $1.05p$ shows that adding a tax is the same as multiplying the price by a constant factor.

Algebraic manipulations are governed by the properties of operations and exponents and the conventions of algebraic notation. At times, an expression is the result of applying operations to simpler expressions. For example, $p + 0.05p$ is the sum of the simpler expressions p and $0.05p$. Viewing an expression as the result of operation on simpler expressions can sometimes clarify its underlying structure. A spreadsheet or a computer algebra system (CAS) can be used to experiment with algebraic expressions, perform complicated algebraic manipulations, and understand how algebraic manipulations behave.

Equations and Inequalities

An equation is a statement of equality between two expressions, often viewed as a question asking for which values of the variables the expressions on either side are in fact equal. These values are the solutions to the equation. An identity, in contrast, is true for all values of the variables; identities are often developed by rewriting an expression in an equivalent form.

The solutions of an equation in one variable form a set of numbers; the solutions of an equation in two variables form a set of ordered pairs of numbers, which can be plotted in the coordinate plane. Two or more equations and/or inequalities form a system. A solution for such a system must satisfy every equation and inequality in the system.

An equation can often be solved by successively deducing from it one or more simpler equations. For example, one can add the same constant to both sides without changing the solutions, but squaring both sides might lead to extraneous solutions. Strategic competence in solving includes looking ahead for productive manipulations and anticipating the nature and number of solutions.

Some equations have no solutions in a given number system, but have a solution in a larger system. For example, the solution of $x + 1 = 0$ is an integer, not a whole number; the solution of $2x + 1 = 0$ is a rational number, not an integer; the solutions of $x^2 - 2 = 0$ are real numbers, not rational numbers; and the solutions of $x^2 + 2 = 0$ are complex numbers, not real numbers.

The same solution techniques used to solve equations can be used to rearrange formulas. For example, the formula for the area of a trapezoid, $A = ({}^{(b1+b2)}/_{2})h$, can be solved for h using the same deductive process. Inequalities can be solved by reasoning about the properties of inequality. Many, but not all, of the properties of equality continue to hold for inequalities and can be useful in solving them.

Connections to Functions and Modeling

Expressions can define functions, and equivalent expressions define the same function. Asking when two functions have the same value for the same input leads to an equation; graphing the two functions allows for finding approximate solutions of the equation. Converting a verbal description to an equation, inequality, or system of these is an essential skill in modeling.

Conceptual Category: Algebra Overview [A]

Seeing Structure in Expressions

- **A.** Interpret the structure of expressions (linear, quadratic, exponential, polynomial, rational).
- **B.** Write expressions in equivalent forms to solve problems.

Arithmetic with Polynomials and Rational Expressions

- **A.** Perform arithmetic operations on polynomials.
- **B.** Understand the relationship between zeros and factors of polynomials.
- **C.** Use polynomial identities to solve problems.
- **D.** Rewrite rational expressions.

Creating Equations

- **A.** Create equations that describe numbers or relationships.

Reasoning with Equations and Inequalities

- **A.** Understand solving equations as a process of reasoning and explain the reasoning.
- **B.** Solve equations and inequalities in one variable.
- **C.** Solve systems of equations.
- **D.** Represent and solve equations and inequalities graphically.

Standards for Mathematical Practice

1. Make sense of problems and persevere in solving them.
2. Reason abstractly and quantitatively.
3. Construct viable arguments and critique the reasoning of others.
4. Model with mathematics.
5. Use appropriate tools strategically.
6. Attend to precision.
7. Look for and make use of structure.
8. Look for and express regularity in repeated reasoning.

Conceptual Category: Algebra Content Standards [A]

Seeing Structure in Expressions A-SSE

A. Interpret the structure of linear, quadratic, exponential, polynomial, and rational expressions.

1. Interpret expressions that represent a quantity in terms of its context.★
    - **a.** Interpret parts of an expression, such as terms, factors, and coefficients.
    - **b.** Interpret complicated expressions by viewing one or more of their parts as a single entity.

    For example, interpret $P(1 + r)^n$ as the product of P and a factor not depending on P.
2. Use the structure of an expression to identify ways to rewrite it.

    For example, see $x^4 - y^4$ as $(x^2)^2 - (y^2)^2$, thus recognizing it as a difference of squares that can be factored as $(x^2 - y^2)(x^2 + y^2)$.

B. Write expressions in equivalent forms to solve problems.

3. Choose and produce an equivalent form of an expression to reveal and explain properties of the quantity represented by the expression.
    - **a.** Factor a quadratic expression to reveal the zeros of the function it defines.
    - **b.** Complete the square in a quadratic expression to reveal the maximum or minimum value of the function it defines.
    - **c.** Use the properties of exponents to transform expressions for exponential functions.

    For example, the expression 1.15^t can be rewritten as $(1.15^{1/12})^{12t} \approx 1.012^{12t}$ to reveal the approximate equivalent monthly interest rate if the annual rate is 15%.
4. Derive the formula for the sum of a finite geometric series (when the common ratio is not 1), and use the formula to solve problems.★

    For example, calculate mortgage payments.

Arithmetic with Polynomials and Rational Expressions A-APR

A. Perform arithmetic operations on polynomials.

1. Understand that polynomials form a system analogous to the integers, namely, they are closed under certain operations.
    - **a.** Perform operations on polynomial expressions (addition, subtraction, multiplication, division) and compare the system of polynomials to the system of integers when performing operations.
    - **b.** Factor and/or expand polynomial expressions, identify and combine like terms, and apply the Distributive property.

B. Understand the relationship between zeros and factors of polynomials.

2. Know and apply the Remainder Theorem: For a polynomial $p(x)$ and a number a, the remainder on division by $x - a$ is $p(a)$, so $p(a) = 0$ if and only if $(x - a)$ is a factor of $p(x)$.
3. Identify zeros of polynomials when suitable factorizations are available, and use the zeros to construct a rough graph of the function defined by the polynomial.

C. Use polynomial identities to solve problems.

4. Prove polynomial identities and use them to describe numerical relationships.

    For example, the polynomial identity $(x^2 + y^2)^2 = (x^2 - y^2)^2 + (2xy)^2$ can be used to generate Pythagorean triples.
5. (+) Know and apply the Binomial Theorem for the expansion of $(x + y)^n$ in powers of x and y for a positive integer n, where x and y are any numbers, with coefficients determined for example by Pascal's Triangle.

D. Rewrite rational expressions.

6. Rewrite simple rational expressions in different forms; write ${}^{a(x)}/_{b(x)}$ in the form $q(x) + {}^{r(x)}/_{b(x)}$, where $a(x)$, $b(x)$, $q(x)$, and $r(x)$ are polynomials with the degree of $r(x)$ less than the degree of $b(x)$, using inspection, long division, or, for the more complicated examples, a computer algebra system.
7. (+) Understand that rational expressions form a system analogous to the rational numbers, closed under addition, subtraction, multiplication, and division by a nonzero rational expression; add, subtract, multiply, and divide rational expressions.

Creating Equations A-CED

A. Create equations that describe numbers or relationships.

1. Create equations and inequalities in one variable and use them to solve problems. (Include equations arising from linear and quadratic functions, and simple root and rational functions and exponential functions.)★
2. Create equations in two or more variables to represent relationships between quantities; graph equations on coordinate axes with labels and scales.★
3. Represent constraints by equations or inequalities, and by systems of equations and/or inequalities, and interpret solutions as viable or non-viable options in a modeling context.★

   For example, represent inequalities describing nutritional and cost constraints on combinations of different foods.
4. Rearrange formulas to highlight a quantity of interest, using the same reasoning as in solving equations. ★

   For example, rearrange Ohm's law $V = IR$ to highlight resistance, R.

Reasoning with Equations and Inequalities A-REI

A. Understand solving equations as a process of reasoning and explain the reasoning.

1. Explain each step in solving a simple equation as following from the equality of numbers asserted at the previous step, starting from the assumption that the original equation has a solution. Construct a viable argument to justify or refute a solution method.
2. Solve simple rational and radical equations in one variable, and give examples showing how extraneous solutions may arise.

B. Solve equations and inequalities in one variable.

3. Solve linear equations and inequalities in one variable, including equations with coefficients represented by letters.
   - **a.** Solve linear equations and inequalities in one variable involving absolute value.
4. Solve quadratic equations in one variable.
   - **a.** Use the method of completing the square to transform any quadratic equation in x into an equation of the form $(x – p)^2 = q$ that has the same solutions. Derive the quadratic formula from this form.
   - **b.** Solve quadratic equations by inspection (e.g., for $x^2 = 49$), taking square roots, completing the square, the quadratic formula, and factoring, as appropriate to the initial form of the equation. Recognize when the quadratic formula gives complex solutions and write them as $a \pm bi$ for real numbers a and b.

C. Solve systems of equations.

5. Prove that, given a system of two equations in two variables, replacing one equation by the sum of that equation and a multiple of the other produces a system with the same solutions.
6. Solve systems of linear equations exactly and approximately (e.g., with graphs), focusing on pairs of linear equations in two variables.
7. Solve a simple system consisting of a linear equation and a quadratic equation in two variables algebraically and graphically.

   For example, find the points of intersection between the line $y = –3x$ and the circle $x^2 + y^2 = 3$.
8. (+) Represent a system of linear equations as a single matrix equation in a vector variable.
9. (+) Find the inverse of a matrix if it exists and use it to solve systems of linear equations (using technology for matrices of dimension 3×3 or greater).

D. Represent and solve equations and inequalities graphically.

10. Understand that the graph of an equation in two variables is the set of all its solutions plotted in the coordinate plane, often forming a curve (which could be a line). Show that any point on the graph of an equation in two variables is a solution to the equation.
11. Explain why the x-coordinates of the points where the graphs of the equations $y = f(x)$ and $y = g(x)$ intersect are the solutions of the equation $f(x) = g(x)$; find the solutions approximately, e.g., using technology to graph the functions, make tables of values, or find successive approximations. Include cases where $f(x)$ and/or $g(x)$ are linear, polynomial, rational, absolute value, exponential, and logarithmic functions.★

12. Graph the solutions of a linear inequality in two variables as a half-plane (excluding the boundary in the case of a strict inequality), and graph the solution set of a system of linear inequalities in two variables as the intersection of the corresponding half-planes.

Conceptual Category: Functions [F]

Introduction

Functions describe situations where one quantity determines another. For example, the return on $10,000 invested at an annualized percentage rate of 4.25% is a function of the length of time the money is invested. Because we continually make theories about dependencies between quantities in nature and society, functions are important tools in the construction of mathematical models.

In school mathematics, functions usually have numerical inputs and outputs and are often defined by an algebraic expression. For example, the time in hours it takes for a car to drive 100 miles is a function of the car's speed in miles per hour, v; the rule $T(v) = {}^{100}/_{v}$ expresses this relationship algebraically and defines a function whose name is T.

The set of inputs to a function is called its domain. We often infer the domain to be all inputs for which the expression defining a function has a value, or for which the function makes sense in a given context.

A function can be described in various ways, such as; by a graph (e.g., the trace of a seismograph); by a verbal rule, as in, "I'll give you a state, you give me the capital city"; by an algebraic expression like $f(x) = a + bx$; or by a recursive rule. The graph of a function is often a useful way of visualizing the relationship of the function models, and manipulating a mathematical expression for a function can throw light on the function's properties.

Functions presented as expressions can model many important phenomena. Two important families of functions characterized by laws of growth are linear functions, which grow at a constant rate, and exponential functions, which grow at a constant percent rate. Linear functions with a constant term of zero describe proportional relationships.

A graphing utility or a computer algebra system can be used to experiment with properties of these functions and their graphs and to build computational models of functions, including recursively defined functions.

Connections to Expressions, Equations, Modeling, and Coordinates

Determining an output value for a particular input involves evaluating an expression; finding inputs that yield a given output involves solving an equation. Questions about when two functions have the same value for the same input lead to equations, whose solutions can be visualized from the intersection of their graphs. Because functions describe relationships between quantities, they are frequently used in modeling. Sometimes functions are defined by a recursive process, which can be displayed effectively using a spreadsheet or other technology.

Conceptual Category: Functions Overview [F]

Interpreting Functions

- **A.** Understand the concept of a function and use function notation.
- **B.** Interpret functions that arise in applications in terms of the context (linear, quadratic, exponential, rational, polynomial, square root, cube root, trigonometric, logarithmic).
- **C.** Analyze functions using different representations.

Building Functions

- **A.** Build a function that models a relationship between two quantities.
- **B.** Build new functions from existing functions.

Linear, Quadratic, and Exponential Models

- **A.** Construct and compare linear, quadratic, and exponential models and solve problems.
- **B.** Interpret expressions for functions in terms of the situation they model.

Trigonometric Functions

- **A.** Extend the domain of trigonometric functions using the unit circle.
- **B.** Model periodic phenomena with trigonometric functions.
- **C.** Prove and apply trigonometric identities.

Standards for Mathematical Practice

1. Make sense of problems and persevere in solving them.
2. Reason abstractly and quantitatively.
3. Construct viable arguments and critique the reasoning of others.
4. Model with mathematics.
5. Use appropriate tools strategically.
6. Attend to precision.
7. Look for and make use of structure.
8. Look for and express regularity in repeated reasoning.

Conceptual Category: Functions Content Standards [F]

Interpreting Functions F-IF

A. Understand the concept of a function and use function notation.

1. Understand that a function from one set (called the domain) to another set (called the range) assigns to each element of the domain exactly one element of the range. If f is a function and x is an element of its domain, then $f(x)$ denotes the output of f corresponding to the input x. The graph of f is the graph of the equation $y = f(x)$.
2. Use function notation, evaluate functions for inputs in their domains, and interpret statements that use function notation in terms of a context.

   For example, given a function representing a car loan, determine the balance of the loan at different points in time.
3. Recognize that sequences are functions, sometimes defined recursively, whose domain is a subset of the integers.

   For example, the Fibonacci sequence is defined recursively by $f(0) = f(1) = 1$, $f(n + 1) = f(n) + f(n - 1)$ ***for*** $n \geq 1$.

B. Interpret functions that arise in applications in terms of the context (linear, quadratic, exponential, rational, polynomial, square root, cube root, trigonometric, logarithmic).

4. For a function that models a relationship between two quantities, interpret key features of graphs and tables in terms of the quantities, and sketch graphs showing key features given a verbal description of the relationship. Key features include: intercepts; intervals where the function is increasing, decreasing, positive, or negative; relative maximums and minimums; symmetries; end behavior; and periodicity.★
5. Relate the domain of a function to its graph and, where applicable, to the quantitative relationship it describes.★

   For example, if the function h(n) gives the number of person-hours it takes to assemble n engines in a factory, then the positive integers would be an appropriate domain for the function.
6. Calculate and interpret the average rate of change of a function (presented symbolically or as a table) over a specified interval. Estimate the rate of change from a graph.★

C. Analyze functions using different representations.

7. Graph functions expressed symbolically and show key features of the graph, by hand in simple cases and using technology for more complicated cases.★
   - **a.** Graph linear and quadratic functions and show intercepts, maxima, and minima.★
   - **b.** Graph square root, cube root, and piecewise-defined functions, including step functions and absolute value functions.★
   - **c.** Graph polynomial functions, identifying zeros when suitable factorizations are available, and showing end behavior.★
   - **d.** (+) Graph rational functions, identifying zeros and asymptotes when suitable factorizations are available, and showing end behavior.★
   - **e.** Graph exponential and logarithmic functions, showing intercepts and end behavior, and trigonometric functions, showing period, midline, and amplitude.★
8. Write a function defined by an expression in different but equivalent forms to reveal and explain different properties of the function.
   - **a.** Use the process of factoring and/or completing the square in quadratic and polynomial functions, where appropriate, to show zeros, extreme values, and symmetry of the graph, and interpret these in terms of a context.
   - **b.** Use the properties of exponents to interpret expressions for exponential functions. Apply to financial situations such as identifying appreciation and depreciation rate for the value of a house or car some time after its initial purchase. $V_n = P(1+r)^n$

   For example, identify percent rate of change in functions such as $y = (1.02)^t$, $y = (0.97)^t$, $y = (1.01)^{12t}$, ***and*** $y = (1.2)^{t/10}$, ***and classify them as representing exponential growth or decay.***
9. Translate among different representations of functions (algebraically, graphically, numerically in tables, or by verbal descriptions). Compare properties of two functions each represented in a different way.

For example, given a graph of one polynomial function (including quadratic functions) and an algebraic expression for another, say which has the larger/smaller relative maximum and/or minimum.

10. Given algebraic, numeric and/or graphical representations of functions, recognize the function as polynomial, rational, logarithmic, exponential, or trigonometric.

Building Functions F-BF

A. Build a function that models a relationship between two quantities.

1. Write a function (linear, quadratic, exponential, simple rational, radical, logarithmic, and trigonometric) that describes a relationship between two quantities.★

- **a.** Determine an explicit expression, a recursive process, or steps for calculation from a context.★
- **b.** Combine standard function types using arithmetic operations.★

For example, build a function that models the temperature of a cooling body by adding a constant function to a decaying exponential, and relate these functions to the model.

- **c.** (+) Compose functions.★

For example, if $T(y)$ is the temperature in the atmosphere as a function of height, and $h(t)$ is the height of a weather balloon as a function of time, then $T(h(t))$ is the temperature at the location of the weather balloon as a function of time.

2. Write arithmetic and geometric sequences both recursively and with an explicit formula, use them to model situations, and translate between the two forms.★

B. Build new functions from existing functions.

3. Identify the effect on the graph of replacing $f(x)$ by $f(x) + k$, $kf(x)$, $f(kx)$, and $f(x + k)$ for specific values of k (both positive and negative); find the value of k given the graphs. (Include linear, quadratic, exponential, absolute value, simple rational and radical, logarithmic and trigonometric functions.) Utilize technology to experiment with cases and illustrate an explanation of the effects on the graph. (Include recognizing even and odd functions from their graphs and algebraic expressions for them.)

4. Find inverse functions algebraically and graphically.

- **a.** Solve an equation of the form $f(x) = c$ for a simple function f that has an inverse and write an expression for the inverse. (Include linear and simple polynomial, rational, and exponential functions.)

For example, $f(x) =2x^3$ or $f(x) = {}^{(x+1)}/_{(x-1)}$ for $x \neq 1$.

- **b.** (+) Verify by composition that one function is the inverse of another.
- **c.** (+) Read values of an inverse function from a graph or a table, given that the function has an inverse.
- **d.** (+) Produce an invertible function from a non-invertible function by restricting the domain.

5. (+) Understand the inverse relationship between exponents and logarithms and use this relationship to solve problems involving logarithms and exponents.

Linear, Quadratic, and Exponential Models F-LE

A. Construct and compare linear, quadratic, and exponential models and solve problems.

1. Distinguish between situations that can be modeled with linear functions and with exponential functions.★

- **a.** Prove that linear functions grow by equal differences over equal intervals, and that exponential functions grow by equal factors over equal intervals.★
- **b.** Recognize situations in which one quantity changes at a constant rate per unit interval relative to another.★
- **c.** Recognize situations in which a quantity grows or decays by a constant percent rate per unit interval relative to another.★

2. Construct linear and exponential functions, including arithmetic and geometric sequences, given a graph, a description of a relationship, or two input-output pairs (including reading these from a table).★

3. Observe, using graphs and tables, that a quantity increasing exponentially eventually exceeds a quantity increasing linearly, quadratically, or (more generally) as a polynomial function.★

4. For exponential models, express as a logarithm the solution to $ab^{ct} = d$ where a, c, and d are numbers and the base b is 2, 10, or e; evaluate the logarithm using technology.★

B. Interpret expressions for functions in terms of the situation they model.

5. Interpret the parameters in a linear or exponential function (of the form $f(x) = b^x + k$) in terms of a context.★

Trigonometric Functions F-TF

A. Extend the domain of trigonometric functions using the unit circle.

1. Understand radian measure of an angle as the length of the arc on the unit circle subtended by the angle.
2. Explain how the unit circle in the coordinate plane enables the extension of trigonometric functions to all real numbers, interpreted as radian measures of angles traversed counterclockwise around the unit circle.
3. (+) Use special triangles to determine geometrically the values of sine, cosine, tangent for $^{\pi}/_{3}$, $^{\pi}/_{4}$ and $^{\pi}/_{6}$, and use the unit circle to express the values of sine, cosine, and tangent for $\pi - x$, $\pi + x$, and $2\pi - x$ in terms of their values for x, where x is any real number.
4. (+) Use the unit circle to explain symmetry (odd and even) and periodicity of trigonometric functions.

B. Model periodic phenomena with trigonometric functions.

5. Choose trigonometric functions to model periodic phenomena with specified amplitude, frequency, and midline.★
6. (+) Understand that restricting a trigonometric function to a domain on which it is always increasing or always decreasing allows its inverse to be constructed.
7. (+) Use inverse functions to solve trigonometric equations that arise in modeling contexts; evaluate the solutions using technology, and interpret them in terms of the context.★

C. Prove and apply trigonometric identities.

8. Prove the Pythagorean identity $\sin^2(\theta) + \cos^2(\theta) = 1$ and use it to find $\sin(\theta)$, $\cos(\theta)$, or $\tan(\theta)$ given $\sin(\theta)$, $\cos(\theta)$, or $\tan(\theta)$ and the quadrant.
9. (+) Prove the addition and subtraction formulas for sine, cosine, and tangent and use them to solve problems.

Conceptual Category: Modeling [★]

Introduction

Modeling links classroom mathematics and statistics to everyday life, work, and decision-making. Modeling is the process of choosing and using appropriate mathematics and statistics to analyze empirical situations, to understand them better, and to improve decisions. Quantities and their relationships in physical, economic, public policy, social, and everyday situations can be modeled using mathematical and statistical methods. When making mathematical models, technology is valuable for varying assumptions, exploring consequences, and comparing predictions with data.

A model can be very simple, such as writing total cost as a product of unit price and number bought, or using a geometric shape to describe a physical object like a coin. Even such simple models involve making choices. It is up to us whether to model a coin as a three-dimensional cylinder, or whether a two-dimensional disk works well enough for our purposes. Other situations—modeling a delivery route, a production schedule, or a comparison of loan amortizations—need more elaborate models that use other tools from the mathematical sciences. Real-world situations are not organized and labeled for analysis; formulating tractable models, representing such models, and analyzing them is appropriately a creative process. Like every such process, this depends on acquired expertise as well as creativity.

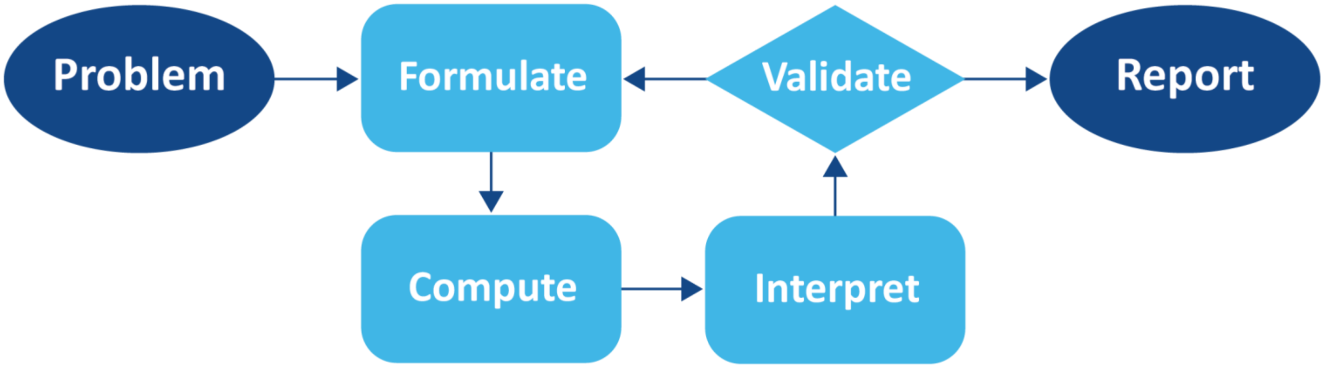
Some examples of such situations might include:

- Estimating how much water and food is needed for emergency relief in a devastated city of three million people, and how it might be distributed.
- Planning a table tennis tournament for seven players at a club with four tables, where each player plays against each other player.
- Designing the layout of the stalls in a school fair so as to raise as much money as possible.
- Analyzing stopping distance for a car.
- Modeling savings account balance, bacterial colony growth, or investment growth.
- Engaging in critical path analysis, e.g., applied to turnaround of an aircraft at an airport.
- Analyzing risk in situations such as extreme sports, pandemics, and terrorism.
- Relating population statistics to individual predictions.

In situations like these, the models devised depend on a number of factors: How precise an answer do we want or need? What aspects of the situation do we most need to understand, control, or optimize? What resources of time and tools do we have? The range of models that we can create and analyze is also constrained by the limitations of our mathematical, statistical, and technical skills, and our ability to recognize significant variables and relationships among them. Diagrams of various kinds, spreadsheets and other technology, and algebra are powerful tools for understanding and solving problems drawn from different types of real-world situations.

One of the insights provided by mathematical modeling is that essentially the same mathematical or statistical structure can sometimes model seemingly different situations. Models can also shed light on the mathematical structures themselves, for example, as when a model of bacterial growth makes more vivid the explosive growth of the exponential function.

The basic modeling cycle is summarized in the diagram below. It involves: (1) identifying variables in the situation and selecting those that represent essential features; (2) formulating a model by creating and selecting geometric, graphical, tabular, algebraic, or statistical representations that describe relationships between the variables; (3) analyzing and performing operations on these relationships to draw conclusions; (4) interpreting the results of the mathematics in terms of the original situation; (5) validating the conclusions by comparing them with the situation, and then either improving the model or; (6) if it is acceptable, reporting on the conclusions and the reasoning behind them. Choices, assumptions, and approximations are present throughout this cycle.



In descriptive modeling, a model simply describes the phenomena or summarizes them in a compact form. Graphs of observations are a familiar descriptive model—for example, graphs of global temperature and atmospheric CO_2 over time.

Analytic modeling seeks to explain data on the basis of deeper theoretical ideas, albeit with parameters that are empirically based; for example, exponential growth of bacterial colonies (until cut-off mechanisms such as pollution or starvation intervene) follows from a constant reproduction rate. Functions are an important tool for analyzing such problems.

Graphing utilities, spreadsheets, computer algebra systems, and dynamic geometry software are powerful tools that can be used to model purely mathematical phenomena (e.g., the behavior of polynomials) as well as physical phenomena.

Modeling Standards

Modeling is best interpreted not as a collection of isolated topics but rather in relation to other standards. Making mathematical models is a Standard for Mathematical Practice, and specific Modeling standards appear throughout the high school standards indicated by a star symbol (★).

Conceptual Category: Geometry [G]

Introduction

An understanding of the attributes and relationships of geometric objects can be applied in diverse contexts—interpreting a schematic drawing, estimating the amount of wood needed to frame a sloping roof, rendering computer graphics, or designing a sewing pattern for the most efficient use of material.

Although there are many types of geometry, school mathematics is devoted primarily to plane Euclidean geometry, studied both synthetically (without coordinates) and analytically (with coordinates). Euclidean geometry is characterized most importantly by the Parallel Postulate that through a point not on a given line there is exactly one parallel line. (Spherical geometry, in contrast, has no parallel lines.)

During high school, students begin to formalize their geometry experiences from elementary and middle school, using more precise definitions and developing careful proofs. Later in college, some students develop Euclidean and other geometries carefully from a small set of axioms.

The concepts of congruence, similarity, and symmetry can be understood from the perspective of geometric transformation. Fundamental are the rigid motions: translations, rotations, reflections, and combinations of these, all of which are here assumed to preserve distance and angles (and therefore shapes generally). Reflections and rotations each explain a particular type of symmetry, and the symmetries of an object offer insight into its attributes—as when the reflective symmetry of an isosceles triangle assures that its base angles are congruent.

In the approach taken here, two geometric figures are defined to be congruent if there is a sequence of rigid motions that carries one onto the other. This is the principle of superposition. For triangles, congruence means the equality of all corresponding pairs of sides and all corresponding pairs of angles. During the middle grades, through experiences drawing triangles from given conditions, students notice ways to specify enough measures in a triangle to ensure that all triangles drawn with those measures are congruent. Once these triangle congruence criteria (ASA, SAS, and SSS) are established using rigid motions, they can be used to prove theorems about triangles, quadrilaterals, and other geometric figures.

Similarity transformations (rigid motions followed by dilations) define similarity in the same way that rigid motions define congruence, thereby formalizing the similarity ideas of “same shape” and “scale factor” developed in the middle grades. These transformations lead to the criterion for triangle similarity that two pairs of corresponding angles are congruent.

The definitions of sine, cosine, and tangent for acute angles are founded on right triangles and similarity, and, with the Pythagorean Theorem, are fundamental in many real-world and theoretical situations. The Pythagorean Theorem is generalized to non-right triangles by the Law of Cosines. Together, the Laws of Sines and Cosines embody the triangle congruence criteria for the cases where three pieces of information suffice to completely solve a triangle. Furthermore, these laws yield two possible solutions in the ambiguous case, illustrating that Side-Side-Angle is not a congruence criterion.

Analytic geometry connects algebra and geometry, resulting in powerful methods of analysis and problem solving. Just as the number line associates numbers with locations in one dimension, a pair of perpendicular axes associates pairs of numbers with locations in two dimensions. This correspondence between numerical coordinates and geometric points allows methods from algebra to be applied to geometry and vice versa. The solution set of an equation becomes a geometric curve, making visualization a tool for doing and understanding algebra. Geometric shapes can be described by equations, making algebraic manipulation into a tool for geometric understanding, modeling, and proof. Geometric transformations of the graphs of equations correspond to algebraic changes in their equations.

Dynamic geometry environments provide students with experimental and modeling tools that allow them to investigate geometric phenomena in much the same way as computer algebra systems allow them to experiment with algebraic phenomena.

Connections to Equations

The correspondence between numerical coordinates and geometric points allows methods from algebra to be applied to geometry and vice versa. The solution set of an equation becomes a geometric curve, making visualization a tool for doing and understanding algebra. Geometric shapes can be described by equations, making algebraic manipulation into a tool for geometric understanding, modeling, and proof.

Conceptual Category: Geometry Overview [G]

Congruence

- **A.** Experiment with transformations in the plane.
- **B.** Understand congruence in terms of rigid motions.
- **C.** Prove geometric theorems and, when appropriate, the converse of theorems.
- **D.** Make geometric constructions.

Similarity, Right Triangles, and Trigonometry

- **A.** Understand similarity in terms of similarity transformations.
- **B.** Prove theorems involving similarity.
- **C.** Define trigonometric ratios and solve problems involving right triangles.
- **D.** Apply trigonometry to general triangles.

Circles

- **A.** Understand and apply theorems about circles.
- **B.** Find arc lengths and areas of sectors of circles.

Expressing Geometric Properties with Equations

- **A.** Translate between the geometric description and the equation for a conic section.
- **B.** Use coordinates to prove simple geometric theorems algebraically.

Standards for Mathematical Practice

1. Make sense of problems and persevere in solving them.
2. Reason abstractly and quantitatively.
3. Construct viable arguments and critique the reasoning of others.
4. Model with mathematics.
5. Use appropriate tools strategically.
6. Attend to precision.
7. Look for and make use of structure.
8. Look for and express regularity in repeated reasoning.

Geometric Measurement and Dimension

- **A.** Explain volume formulas and use them to solve problems.
- **B.** Visualize relationships between two-dimensional and three-dimensional objects.

Modeling with Geometry

- **A.** Apply geometric concepts in modeling situations.

Conceptual Category: Geometry Content Standards [G]

Congruence G-CO

A. Experiment with transformations in the plane.

1. Know precise definitions of angle, circle, perpendicular line, parallel line, and line segment, based on the undefined notions of point, line, distance along a line, and distance around a circular arc.
2. Represent transformations in the plane using, e.g., transparencies and geometry software; describe transformations as functions that take points in the plane as inputs and give other points as outputs. Compare transformations that preserve distance and angle to those that do not (e.g., translation versus horizontal stretch).
3. Given a rectangle, parallelogram, trapezoid, or regular polygon, describe the rotations and reflections that carry it onto itself.
4. Develop definitions of rotations, reflections, and translations in terms of angles, circles, perpendicular lines, parallel lines, and line segments.
5. Given a geometric figure and a rotation, reflection, or translation, draw the transformed figure using, e.g., graph paper, tracing paper, or geometry software. Specify a sequence of transformations that will carry a given figure onto another.

B. Understand congruence in terms of rigid motions.

6. Use geometric descriptions of rigid motions to transform figures and to predict the effect of a given rigid motion on a given figure; given two figures, use the definition of congruence in terms of rigid motions to decide if they are congruent.
7. Use the definition of congruence in terms of rigid motions to show that two triangles are congruent if and only if corresponding pairs of sides and corresponding pairs of angles are congruent.
8. Explain how the criteria for triangle congruence (ASA, SAS, and SSS) follow from the definition of congruence in terms of rigid motions.

C. Prove geometric theorems and, when appropriate, the converse of theorems.

9. Prove theorems about lines and angles. Theorems include: vertical angles are congruent; when a transversal crosses parallel lines, alternate interior angles are congruent and corresponding angles are congruent, and conversely prove lines are parallel; points on a perpendicular bisector of a line segment are exactly those equidistant from the segment's endpoints.
10. Prove theorems about triangles. Theorems include: measures of interior angles of a triangle sum to 180°; base angles of isosceles triangles are congruent, and conversely prove a triangle is isosceles; the segment joining midpoints of two sides of a triangle is parallel to the third side and half the length; and the medians of a triangle meet at a point.
11. Prove theorems about parallelograms. Theorems include: opposite sides are congruent, opposite angles are congruent, the diagonals of a parallelogram bisect each other, and conversely, rectangles are parallelograms with congruent diagonals.
    a. Prove theorems about polygons. Theorems include the measures of interior and exterior angles. Apply properties of polygons to the solutions of mathematical and contextual problems.

D. Make geometric constructions.

12. Make formal geometric constructions with a variety of tools and methods (compass and straightedge, string, reflective devices, paper folding, dynamic geometric software, etc.). Constructions include: copying a segment; copying an angle; bisecting a segment; bisecting an angle; constructing perpendicular lines, including the perpendicular bisector of a line segment; and constructing a line parallel to a given line through a point not on the line.
13. Construct an equilateral triangle, a square, and a regular hexagon inscribed in a circle.

Similarity, Right Triangles, and Trigonometry G-SRT

A. Understand similarity in terms of similarity transformations.

1. Verify experimentally the properties of dilations given by a center and a scale factor:
    - **a.** A dilation takes a line not passing through the center of the dilation to a parallel line, and leaves a line passing through the center unchanged.
    - **b.** The dilation of a line segment is longer or shorter in the ratio given by the scale factor.
2. Given two figures, use the definition of similarity in terms of similarity transformations to decide if they are similar; explain using similarity transformations the meaning of similarity for triangles as the equality of all corresponding pairs of angles and the proportionality of all corresponding pairs of sides.
3. Use the properties of similarity transformations to establish the Angle-Angle (AA) criterion for two triangles to be similar.

B. Prove theorems involving similarity.

4. Prove theorems about triangles. Theorems include: a line parallel to one side of a triangle divides the other two proportionally, and conversely; the Pythagorean Theorem proved using triangle similarity.
5. Use congruence and similarity criteria for triangles to solve problems and to prove relationships in geometric figures.

C. Define trigonometric ratios and solve problems involving right triangles.

6. Understand that by similarity, side ratios in right triangles are properties of the angles in the triangle, leading to definitions of trigonometric ratios for acute angles.
7. Explain and use the relationship between the sine and cosine of complementary angles.
8. Use trigonometric ratios and the Pythagorean Theorem to solve right triangles in applied problems.★

D. Apply trigonometry to general triangles.

9. (+) Derive the formula $A = \frac{1}{2} ab \sin(C)$ for the area of a triangle by drawing an auxiliary line from a vertex perpendicular to the opposite side.
10. (+) Prove the Laws of Sines and Cosines and use them to solve problems.
11. (+) Understand and apply the Law of Sines and the Law of Cosines to find unknown measurements in right and non-right triangles (e.g., surveying problems, resultant forces).

Circles G-C

A. Understand and apply theorems about circles.

1. Prove that all circles are similar.
2. Identify and describe relationships among inscribed angles, radii, and chords. Include the relationship between central, inscribed, and circumscribed angles; inscribed angles on a diameter are right angles; the radius of a circle is perpendicular to the tangent where the radius intersects the circle.
3. Construct the inscribed and circumscribed circles of a triangle, and prove properties of angles for a quadrilateral and other polygons inscribed in a circle.
4. (+) Construct a tangent line from a point outside a given circle to the circle.

B. Find arc lengths and areas of sectors of circles.

5. Derive using similarity the fact that the length of the arc intercepted by an angle is proportional to the radius, and define the radian measure of the angle as the constant of proportionality; derive the formula for the area of a sector.

Expressing Geometric Properties with Equations G-GPE

A. Translate between the geometric description and the equation for a conic section.

1. Derive the equation of a circle of given center and radius using the Pythagorean Theorem; complete the square to find the center and radius of a circle given by an equation.
2. Derive the equation of a parabola given a focus and directrix.
3. (+) Derive the equations of ellipses and hyperbolas given the foci, using the fact that the sum or difference of distances from the foci is constant.

- **a.** (+) Use equations and graphs of conic sections to model real-world problems.★

B. Use coordinates to prove simple geometric theorems algebraically.

4. Use coordinates to prove simple geometric theorems algebraically including the distance formula and its relationship to the Pythagorean Theorem.
For example, prove or disprove that a figure defined by four given points in the coordinate plane is a rectangle; prove or disprove that the point (1,$\sqrt{3}$) lies on the circle centered at the origin and containing the point (0, 2).

5. Prove the slope criteria for parallel and perpendicular lines and use them to solve geometric problems (e.g., find the equation of a line parallel or perpendicular to a given line that passes through a given point).

6. Find the point on a directed line segment between two given points that partitions the segment in a given ratio.

7. Use coordinates to compute perimeters of polygons and areas of triangles and rectangles (e.g., using the distance formula).★

Geometric Measurement and Dimension G-GMD

A. Explain volume formulas and use them to solve problems.

1. Give an informal argument for the formulas for the circumference of a circle, area of a circle, volume of a cylinder, pyramid, and cone. *Use dissection arguments, Cavalieri's principle, and informal limit arguments.*

2. (+) Give an informal argument using Cavalieri's principle for the formulas for the volume of a sphere and other solid figures.

3. Use volume formulas for cylinders, pyramids, cones, and spheres to solve problems.★

B. Visualize relationships between two-dimensional and three-dimensional objects.

4. Identify the shapes of two-dimensional cross sections of three-dimensional objects, and identify three-dimensional objects generated by rotations of two-dimensional objects.

Modeling with Geometry G-MG

A. Apply geometric concepts in modeling situations.

1. Use geometric shapes, their measures, and their properties to describe objects (e.g., modeling a tree trunk or a human torso as a cylinder).★

2. Apply concepts of density based on area and volume in modeling situations (e.g., persons per square mile, BTUs per cubic foot).★

3. Apply geometric methods to solve design problems (e.g., designing an object or structure to satisfy physical constraints or minimize cost; working with typographic grid systems based on ratios).★

4. Use dimensional analysis for unit conversions to confirm that expressions and equations make sense.★

Conceptual Category: Statistics and Probability [S]

Introduction

Decisions or predictions are often based on data—numbers in context. These decisions or predictions would be easy if the data always sent a clear message, but the message is often obscured by variability. Statistics provides tools for describing variability in data and for making informed decisions that take it into account.

Data are gathered, displayed, summarized, examined, and interpreted to discover patterns and deviations from patterns. Quantitative data can be described in terms of key characteristics: measures of shape, center, and spread. The shape of a data distribution might be described as symmetric, skewed, flat, or bell shaped, and it might be summarized by a statistic measuring center (such as mean or median) and a statistic measuring spread (such as standard deviation or interquartile range). Different distributions can be compared numerically using these statistics or compared visually using plots. Knowledge of center and spread are not enough to describe a distribution. Which statistics to compare, which plots to use, and what the results of a comparison might mean, depend on the question to be investigated and the real-life actions to be taken.

Randomization has two important uses in drawing statistical conclusions. First, collecting data from a random sample of a population makes it possible to draw valid conclusions about the whole population, taking variability into account. Second, randomly assigning individuals to different treatments allows a fair comparison of the effectiveness of those treatments. A statistically significant outcome is one that is unlikely to be due to chance alone, and this can be evaluated only under the condition of randomness. The conditions under which data are collected are important in drawing conclusions from the data. In critically reviewing uses of statistics in public media and other reports, it is important to consider the study design, how the data were gathered, and the analyses employed, as well as the data summaries and the conclusions drawn.

Random processes can be described mathematically by using a probability model: a list or description of the possible outcomes (the sample space), each of which is assigned a probability. In situations such as flipping a coin, rolling a number cube, or drawing a card, it might be reasonable to assume various outcomes are equally likely. In a probability model, sample points represent outcomes and combine to make up events; probabilities of events can be computed by applying the Addition and Multiplication Rules. Interpreting these probabilities relies on an understanding of independence and conditional probability, which can be approached through the analysis of two-way tables.

Technology plays an important role in statistics and probability by making it possible to generate plots, regression functions, and correlation coefficients, and to simulate many possible outcomes in a short amount of time.

Connections to Functions and Modeling

Functions may be used to describe data; if the data suggest a linear relationship, the relationship can be modeled with a regression line, and its strength and direction can be expressed through a correlation coefficient.

Conceptual Category: Statistics and Probability Overview [S]

Interpreting Categorical and Quantitative Data

- **A.** Summarize, represent, and interpret data on a single count or measurement variable. Use calculators, spreadsheets, and other technology as appropriate.
- **B.** Summarize, represent, and interpret data on two categorical and quantitative variables.
- **C.** Interpret linear models.

Making Inferences and Justifying Conclusions

- **A.** Understand and evaluate random processes underlying statistical experiments. Use calculators, spreadsheets, and other technology as appropriate.
- **B.** Make inferences and justify conclusions from sample surveys, experiments, and observational studies.

Conditional Probability and the Rules of Probability

- **A.** Understand independence and conditional probability and use them to interpret data from simulations or experiments.
- **B.** Use the rules of probability to compute probabilities of compound events in a uniform probability model.

Using Probability to Make Decisions

- **A.** Calculate expected values and use them to solve problems.
- **B.** Use probability to evaluate outcomes of decisions.

Standards for Mathematical Practice

1. Make sense of problems and persevere in solving them.
2. Reason abstractly and quantitatively.
3. Construct viable arguments and critique the reasoning of others.
4. Model with mathematics.
5. Use appropriate tools strategically.
6. Attend to precision.
7. Look for and make use of structure.
8. Look for and express regularity in repeated reasoning.

Conceptual Category: Statistics and Probability Content Standards [S]

Interpreting Categorical and Quantitative Data S-ID

A. Summarize, represent, and interpret data on a single count or measurement variable. Use calculators, spreadsheets, and other technology as appropriate.

1. Represent data with plots on the real number line (dot plots, histograms, and box plots).★
2. Use statistics appropriate to the shape of the data distribution to compare center (median, mean) and spread (interquartile range, standard deviation) of two or more different data sets.★
3. Interpret differences in shape, center, and spread in the context of the data sets, accounting for possible effects of extreme data points (outliers).★
4. Use the mean and standard deviation of a data set to fit it to a normal distribution and to estimate population percentages. Recognize that there are data sets for which such a procedure is not appropriate. Use calculators, spreadsheets, and tables to estimate areas under the normal curve.★

B. Summarize, represent, and interpret data on two categorical and quantitative variables.

5. Summarize categorical data for two categories in two-way frequency tables. Interpret relative frequencies in the context of the data (including joint, marginal, and conditional relative frequencies). Recognize possible associations and trends in the data.★
6. Represent data on two quantitative variables on a scatter plot, and describe how the variables are related.★
    - **a.** Fit a linear function to the data and use the fitted function to solve problems in the context of the data. Use functions fitted to data or choose a function suggested by the context. Emphasize linear and exponential models.★
    - **b.** Informally assess the fit of a function by plotting and analyzing residuals.★
    - **c.** Fit a linear function for a scatter plot that suggests a linear association.★

C. Interpret linear models.

7. Interpret the slope (rate of change) and the intercept (constant term) of a linear model in the context of the data.★
8. Compute (using technology) and interpret the correlation coefficient of a linear fit.★
9. Distinguish between correlation and causation.★

Making Inferences and Justifying Conclusions S-IC

A. Understand and evaluate random processes underlying statistical experiments. Use calculators, spreadsheets, and other technology as appropriate.

1. Understand statistics as a process for making inferences about population parameters based on a random sample from that population.★
2. Decide if a specified model is consistent with results from a given data-generating process (e.g., using simulation).★

    For example, a model says a spinning coin falls heads up with probability 0.5. Would a result of five tails in a row cause you to question the model?

B. Make inferences and justify conclusions from sample surveys, experiments, and observational studies.

3. Recognize the purposes of and differences among sample surveys, experiments, and observational studies; explain how randomization relates to each.★
4. Use data from a sample survey to estimate a population mean or proportion; develop a margin of error through the use of simulation models for random sampling.★
5. Use data from a randomized experiment to compare two treatments; use simulations to decide if differences between parameters are significant.★
6. Evaluate reports based on data.★

Conditional Probability and the Rules of Probability S-CP

A. Understand independence and conditional probability and use them to interpret data from simulations or experiments.

1. Describe events as subsets of a sample space (the set of outcomes) using characteristics (or categories) of the outcomes, or as unions, intersections, or complements of other events ("or," "and," "not").★
2. Understand that two events *A* and *B* are independent if the probability of *A* and *B* occurring together is the product of their probabilities, and use this characterization to determine if they are independent.★
3. Understand the conditional probability of *A* given *B* as *P*(*A* and *B*)/*P*(*B*), and interpret independence of *A* and *B* as saying that the conditional probability of *A* given *B* is the same as the probability of *A*, and the conditional probability of *B* given *A* is the same as the probability of *B*.★
4. Construct and interpret two-way frequency tables of data when two categories are associated with each object being classified. Use the two-way table as a sample space to decide if events are independent and to approximate conditional probabilities.★

   For example, collect data from a random sample of students in your school on their favorite subject among math, science, and English. Estimate the probability that a randomly selected student from your school will favor science given that the student is in tenth grade. Do the same for other subjects and compare the results.
5. Recognize and explain the concepts of conditional probability and independence in everyday language and everyday situations.★

   For example, compare the chance of having lung cancer if you are a smoker with the chance of being a smoker if you have lung cancer.

B. Use the rules of probability to compute probabilities of compound events in a uniform probability model.

6. Find the conditional probability of *A* given *B* as the fraction of *B*'s outcomes that also belong to *A*, and interpret the answer in terms of the model.★
7. Apply the Addition Rule, *P*(*A* or *B*) = *P*(*A*) + *P*(*B*) – *P*(*A* and *B*), and interpret the answer in terms of the model.★
8. (+) Apply the general Multiplication Rule in a uniform probability model, *P*(*A* and *B*) = *P*(*A*)*P*(*B*|*A*) = *P*(*B*)*P*(*A*|*B*), and interpret the answer in terms of the model.★
9. (+) Use permutations and combinations to compute probabilities of compound events and solve problems.★

Using Probability to Make Decisions S-MD

A. Calculate expected values and use them to solve problems.

1. (+) Define a random variable for a quantity of interest by assigning a numerical value to each event in a sample space; graph the corresponding probability distribution using the same graphical displays as for data distributions.★
2. (+) Calculate the expected value of a random variable; interpret it as the mean of the probability distribution.★
3. (+) Develop a probability distribution for a random variable defined for a sample space in which theoretical probabilities can be calculated; find the expected value.★

   For example, find the theoretical probability distribution for the number of correct answers obtained by guessing on all five questions of a multiple-choice test where each question has four choices, and find the expected grade under various grading schemes.
4. (+) Develop a probability distribution for a random variable defined for a sample space in which probabilities are assigned empirically; find the expected value.★

   For example, find a current data distribution on the number of TV sets per household in the United States, and calculate the expected number of sets per household. How many TV sets would you expect to find in 100 randomly selected households?

B. Use probability to evaluate outcomes of decisions.

5. (+) Weigh the possible outcomes of a decision by assigning probabilities to payoff values and finding expected values.★
   - **a.** (+) Find the expected payoff for a game of chance.★

For example, find the expected winnings from a state lottery ticket or a game at a fast-food restaurant.

b. (+) Evaluate and compare strategies on the basis of expected values.★

For example, compare a high-deductible versus a low-deductible automobile insurance policy using various, but reasonable, chances of having a minor or a major accident.

6. (+) Use probabilities to make fair decisions (e.g., drawing by lots or using a random number generator).★

7. (+) Analyze decisions and strategies using probability concepts (e.g., product testing, medical testing, or pulling a hockey goalie at the end of a game and replacing the goalie with an extra skater).★

The Standards for Mathematical Content
High School: Model Pathways and Model Courses

Introduction

Organization of the Model High School Courses

Each model high school course is presented in three sections:

An introduction and description of the critical areas for learning in that course

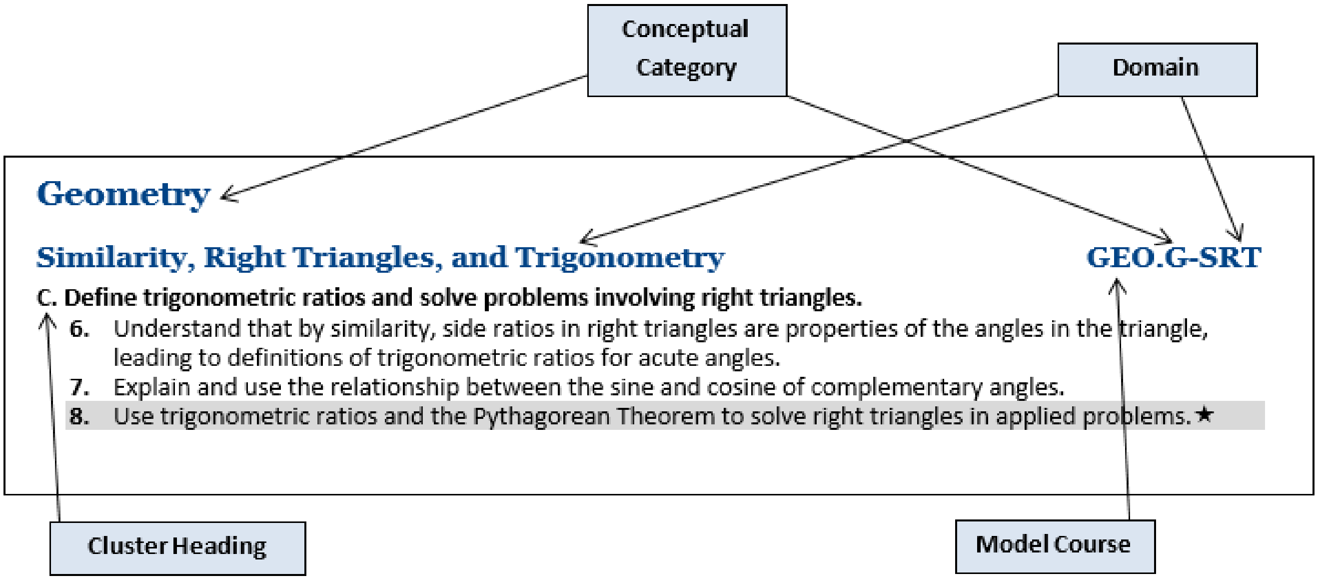
An overview listing the conceptual categories, domains, and clusters included in that course

The content standards for that course, presented by conceptual category, domain, and cluster

Content Standards by Model Course Identifiers/Coding

Standard numbering in the high school model courses is similar to the coding of the high school standards by conceptual category; in addition, a course code has been added at the beginning of each standard to identify the standards in each model course.

The illustration below shows a section of content standards from the Traditional Pathway Model Geometry course. The standard highlighted in the illustration is Standard GEO.G-SRT.C.8, identifying it as a standard from the Geometry Model Course; the Geometry conceptual category (G-), in the Similarity, Right Triangles, and Trigonometry domain (SRT), and as the eighth standard in that domain. The star (★) at the end of the standard indicates that it is a Modeling standard.



Geometry

Similarity, Right Triangles, and Trigonometry **GEO.G-SRT**

C. Define trigonometric ratios and solve problems involving right triangles.

6. Understand that by similarity, side ratios in right triangles are properties of the angles in the triangle, leading to definitions of trigonometric ratios for acute angles.
7. Explain and use the relationship between the sine and cosine of complementary angles.
8. Use trigonometric ratios and the Pythagorean Theorem to solve right triangles in applied problems.★

The star symbol (★) following the standard in the illustration indicates that it is also a Modeling standard. Modeling is best interpreted not as a collection of isolated topics but in relation to other standards. Making mathematical models is a Standard for Mathematical Practice, and specific modeling standards appear throughout the high school standards indicated by a star symbol (★).

Importance of Modeling in High School

Modeling (indicated by a ★ at the end of a standard) is defined as both a *conceptual category* for high school mathematics and a *Standard for Mathematical Practice*, and is an important avenue for motivating students to study mathematics, for building their understanding of mathematics, and for preparing them for future success. Development of the Model Pathways into instructional programs will require careful attention to modeling and the mathematical practices. Assessments based on these Model Pathways should reflect both the Standards for Mathematical Content and the Standards for Mathematical Practice.

The following Model Pathways and Model Courses are presented in this Section:

- **Model Traditional Pathway**
    - **Model Algebra I**
    - **Model Geometry**
    - **Model Algebra II**
- **Model Integrated Pathway**
    - **Model Mathematics I**
    - **Model Mathematics II**
    - **Model Mathematics III**
- **Advanced Model Courses**
    - **Model Precalculus**
    - **Model Advanced Quantitative Reasoning**

Model Traditional Pathway: Model Algebra I [AI]

Introduction

The fundamental purpose of the Model Algebra I course is to formalize and extend the mathematics that students learned in the middle grades. For the high school Model Algebra I course, instructional time should focus on four critical areas: (1) deepen and extend understanding of linear and exponential relationships; (2) contrast linear and exponential relationships with each other and engage in methods for analyzing, solving, and using quadratic functions; (3) extend the laws of exponents to square and cube roots; and (4) apply linear models to data that exhibit a linear trend.

1. By the end of eighth grade, students have learned to solve linear equations in one variable and have applied graphical and algebraic methods to analyze and solve systems of linear equations in two variables. In Algebra I, students analyze and explain the process of solving an equation and justify the process used in solving a system of equations. Students develop fluency writing, interpreting, and translating among various forms of linear equations and inequalities, and use them to solve problems. They master the solution of linear equations and apply related solution techniques and the laws of exponents to the creation and solution of simple exponential equations.
2. In earlier grades, students define, evaluate, and compare functions, and use them to model relationships between quantities. In Algebra I, students learn function notation and develop the concepts of domain and range. They focus on linear, quadratic, and exponential functions, including sequences, and also explore absolute value, step, and piecewise-defined functions; they interpret functions given graphically, numerically, symbolically, and verbally; translate between representations; and understand the limitations of various representations. Students build on and extend their understanding of integer exponents to consider exponential functions. They compare and contrast linear and exponential functions, distinguishing between additive and multiplicative change. Students explore systems of equations and inequalities, and they find and interpret their solutions. They interpret arithmetic sequences as linear functions and geometric sequences as exponential functions.
3. Students extend the laws of exponents to rational exponents involving square and cube roots and apply this new understanding of number; they strengthen their ability to see structure in and create quadratic and exponential expressions. They create and solve equations, inequalities, and systems of equations involving quadratic expressions. Students become facile with algebraic manipulation, including rearranging and collecting terms and factoring. Students consider quadratic functions, comparing the key characteristics of quadratic functions to those of linear and exponential functions. They select from among these functions to model phenomena. Students learn to anticipate the graph of a quadratic function by interpreting various forms of quadratic expressions. In particular, they identify the real solutions of a quadratic equation as the zeros of a related quadratic function. Students expand their experience with functions to include more specialized functions—absolute value, step, and those that are piecewise-defined.
4. Building upon their prior experiences with data, students explore a more formal means of assessing how a model fits data. Students use regression techniques to describe approximately linear relationships between quantities. They use graphical representations and knowledge of context to make judgments about the appropriateness of linear models. With linear models, they look at residuals to analyze the goodness of fit.

The Standards for Mathematical Practice complement the content standards so that students increasingly engage with the subject matter as they grow in mathematical maturity and expertise throughout the elementary, middle, and high school years.

Model Traditional Pathway: Model Algebra I Overview [AI]

Number and Quantity

The Real Number System

- **A.** Extend the properties of exponents to rational exponents.
- **B.** Use properties of rational and irrational numbers.

Quantities

- **A.** Reason quantitatively and use units to solve problems.

Algebra

Seeing Structure in Expressions

- **A.** Interpret the structure of linear, quadratic, and exponential expressions with integer exponents.
- **B.** Write expressions in equivalent forms to solve problems.

Arithmetic with Polynomials and Rational Expressions

- **A.** Perform arithmetic operations on polynomials.

Creating Equations

- **A.** Create equations that describe numbers or relationships.

Reasoning with Equations and Inequalities

- **A.** Understand solving equations as a process of reasoning and explain the reasoning.
- **B.** Solve equations and inequalities in one variable.
- **C.** Solve systems of equations.
- **D.** Represent and solve equations and inequalities graphically.

Standards for Mathematical Practice

1. Make sense of problems and persevere in solving them.
2. Reason abstractly and quantitatively.
3. Construct viable arguments and critique the reasoning of others.
4. Model with mathematics.
5. Use appropriate tools strategically.
6. Attend to precision.
7. Look for and make use of structure.
8. Look for and express regularity in repeated reasoning.

Functions

Interpreting Functions

- **A.** Understand the concept of a function and use function notation.
- **B.** Interpret linear, quadratic, and exponential functions with integer exponents that arise in applications in terms of the context.
- **C.** Analyze functions using different representations.

Building Functions

- **A.** Build a function that models a relationship between two quantities.
- **B.** Build new functions from existing functions.

Linear, Quadratic, and Exponential Models

- **A.** Construct and compare linear, quadratic, and exponential models and solve problems.
- **B.** Interpret expressions for functions in terms of the situation they model.

Statistics and Probability

Interpreting Categorical and Quantitative Data

- **A.** Summarize, represent, and interpret data on a single count or measurement variable. Use calculators, spreadsheets, and other technology as appropriate.
- **B.** Summarize, represent, and interpret data on two categorical and quantitative variables.
- **C.** Interpret linear models.

Model Traditional Pathway: Model Algebra I Content Standards [AI]

Number and Quantity

The Real Number System AI.N-RN

A. Extend the properties of exponents to rational exponents.

1. Explain how the definition of the meaning of rational exponents follows from extending the properties of integer exponents to those values, allowing for a notation for radicals in terms of rational exponents.
   For example, we define $5^{1/3}$ to be the cube root of 5 because we want $(5^{1/3})^3 = 5(^{1/3})^3$ to hold, so $(5^{1/3})^3$ must equal 5.
2. Rewrite expressions involving radicals and rational exponents using the properties of exponents.

B. Use properties of rational and irrational numbers.

3. Explain why the sum or product of two rational numbers is rational; that the sum of a rational number and an irrational number is irrational; and that the product of a nonzero rational number and an irrational number is irrational.

Quantities AI.N-Q

A. Reason quantitatively and use units to solve problems.

1. Use units as a way to understand problems and to guide the solution of multi-step problems; choose and interpret units consistently in formulas; choose and interpret the scale and the origin in graphs and data displays.★
2. Define appropriate quantities for the purpose of descriptive modeling.★
3. Choose a level of accuracy appropriate to limitations on measurement when reporting quantities.★

Algebra

Seeing Structure in Expressions AI.A-SSE

A. Interpret the structure of linear, quadratic, and exponential expressions with integer exponents.

1. Interpret expressions that represent a quantity in terms of its context.★
   - **a.** Interpret parts of an expression, such as terms, factors, and coefficients.
   - **b.** Interpret complicated expressions by viewing one or more of their parts as a single entity.

   For example, interpret $P(1 + r)^t$ as the product of P and a factor not depending on P.
2. Use the structure of an expression to identify ways to rewrite it.
   For example, see $(x + 2)^2 – 9$ as a difference of squares that can be factored as $((x + 2) + 3)((x + 2) – 3)$.

B. Write expressions in equivalent forms to solve problems.

3. Choose and produce an equivalent form of an expression to reveal and explain properties of the quantity represented by the expression.
   - **a.** Factor a quadratic expression to reveal the zeros of the function it defines.
   - **b.** Complete the square in a quadratic expression to reveal the maximum or minimum value of the function it defines.
   - **c.** Use the properties of exponents to transform expressions for exponential functions.

   For example, the expression 1.15^t can be rewritten as $(1.15^{1/12})^{12t} \approx 1.012^{12t}$ to reveal the approximate equivalent monthly interest rate if the annual rate is 15%.

Arithmetic with Polynomials and Rational Expressions AI.A-APR

A. Perform arithmetic operations on polynomials.

1. Understand that polynomials form a system analogous to the integers, namely, they are closed under certain operations.

  - **a.** Perform operations on polynomial expressions (addition, subtraction, multiplication), and compare the system of polynomials to the system of integers when performing operations.
  - **b.** Factor and/or expand polynomial expressions, identify and combine like terms, and apply the Distributive property.

Creating Equations AI.A-CED

A. Create equations that describe numbers or relationships.

1. Create equations and inequalities in one variable and use them to solve problems. (Include equations arising from linear, quadratic, and exponential functions with integer exponents.)★
2. Create equations in two or more variables to represent relationships between quantities; graph equations on coordinate axes with labels and scales.★
3. Represent constraints by linear equations or inequalities, and by systems of linear equations and/or inequalities, and interpret solutions as viable or non-viable options in a modeling context.★

   For example, represent inequalities describing nutritional and cost constraints on combinations of different foods.
4. Rearrange formulas to highlight a quantity of interest using the same reasoning as in solving equations (Properties of equality).★

   For example, rearrange Ohm's law $R=\frac{V^2}{P}$ to solve for voltage, V. Manipulate variables in formulas used in financial contexts such as for simple interest, $I=Prt$.

Reasoning with Equations and Inequalities AI.A-REI

A. Understand solving equations as a process of reasoning and explain the reasoning.

1. Explain each step in solving a simple equation as following from the equality of numbers asserted at the previous step, starting from the assumption that the original equation has a solution. Construct a viable argument to justify or refute a solution method.

B. Solve equations and inequalities in one variable.

3. Solve linear equations and inequalities in one variable, including equations with coefficients represented by letters.
   - **a.** Solve linear equations and inequalities in one variable involving absolute value.
4. Solve quadratic equations in one variable.
   - **a.** Use the method of completing the square to transform any quadratic equation in x into an equation of the form $(x – p)^2 = q$ that has the same solutions. Derive the quadratic formula from this form.
   - **b.** Solve quadratic equations by inspection (e.g., for $x^2 = 49$), taking square roots, completing the square, the quadratic formula, and factoring, as appropriate to the initial form of the equation. Recognize when the solutions of a quadratic equation results in non-real solutions and write them as $a \pm bi$ for real numbers a and b.

C. Solve systems of equations.

5. Prove that, given a system of two equations in two variables, replacing one equation by the sum of that equation and a multiple of the other produces a system with the same solutions.
6. Solve systems of linear equations exactly and approximately (e.g., with graphs), focusing on pairs of linear equations in two variables.
7. Solve a simple system consisting of a linear equation and a quadratic equation in two variables algebraically and graphically.

   For example, find the points of intersection between the line $y = –3x$ and the circle $x^2 + y^2 = 3$.

D. Represent and solve equations and inequalities graphically.

10. Understand that the graph of an equation in two variables is the set of all its solutions plotted in the coordinate plane, often forming a curve (which could be a line). Show that any point on the graph of an equation in two variables is a solution to the equation.
11. Explain why the x-coordinates of the points where the graphs of the equations $y = f(x)$ and $y = g(x)$ intersect are the solutions of the equation $f(x) = g(x)$; find the solutions approximately, e.g., using

technology to graph the functions and make tables of values. Include cases where $f(x)$ and/or $g(x)$ are linear and exponential functions.★

12. Graph the solutions of a linear inequality in two variables as a half-plane (excluding the boundary in the case of a strict inequality), and graph the solution set of a system of linear inequalities in two variables as the intersection of the corresponding half-planes.

Functions

Interpreting Functions AI.F-IF

A. Understand the concept of a function and use function notation.

1. Understand that a function from one set (called the domain) to another set (called the range) assigns to each element of the domain exactly one element of the range. If f is a function and x is an element of its domain, then $f(x)$ denotes the output (range) of f corresponding to the input x. The graph of f is the graph of the equation $y = f(x)$.
2. Use function notation, evaluate functions for inputs in their domains, and interpret statements that use function notation in terms of a context.

   For example, given a function representing a car loan, determine the balance of the loan at different points in time.
3. Recognize that sequences are functions, sometimes defined recursively, whose domain is a subset of the integers.

   For example, the Fibonacci sequence is defined recursively by $f(0) = f(1) = 1$, $f(n + 1) = f(n) + f(n - 1)$ for $n \geq 1$.

B. Interpret linear, quadratic, and exponential functions with integer exponents that arise in applications in terms of the context.

4. For a function that models a relationship between two quantities, interpret key features of graphs and tables in terms of the quantities, and sketch graphs showing key features given a verbal description of the relationship. Key features include: intercepts; intervals where the function is increasing, decreasing, positive, or negative; relative maximums and minimums; symmetries; and end behavior.★
5. Relate the domain of a function to its graph and, where applicable, to the quantitative relationship it describes.★

   For example, if the function h(n) gives the number of person-hours it takes to assemble n engines in a factory, then the positive integers would be an appropriate domain for the function.
6. Calculate and interpret the average rate of change of a function (presented symbolically or as a table) over a specified interval. Estimate the rate of change from a graph.★

C. Analyze functions using different representations.

7. Graph functions expressed symbolically and show key features of the graph, by hand in simple cases and using technology for more complicated cases.★
   - **a.** Graph linear and quadratic functions and show intercepts, maxima, and minima.★
   - **b.** Graph piecewise-defined functions, including step functions and absolute value functions.★
   - **e.** Graph exponential functions showing intercepts and end behavior.★
8. Write a function defined by an expression in different but equivalent forms to reveal and explain different properties of the function.
   - **a.** Use the process of factoring and completing the square in a quadratic function to show zeros, maximum/minimum values, and symmetry of the graph, and interpret these in terms of a context.
   - **b.** Use the properties of exponents to interpret expressions for exponential functions. Apply to financial situations such as identifying appreciation and depreciation rate for the value of a house or car some time after its initial purchase: $V_n = P(1+r)^n$.

   For example, identify percent rate of change in functions such as $y = (1.02)^t$, $y = (0.97)^t$, $y = (1.01)^{12t}$, and $y = (1.2)^{t/10}$, and classify them as representing exponential growth or decay.
9. Translate among different representations of functions (algebraically, graphically, numerically in tables, or by verbal descriptions). Compare properties of two functions each represented in a different way.

For example, given a graph of one quadratic function and an algebraic expression for another, say which has the larger maximum.

Building Functions AI.F-BF

A. Build a function that models a relationship between two quantities.

1. Write linear, quadratic, and exponential functions that describe a relationship between two quantities.★
    - **a.** Determine an explicit expression, a recursive process, or steps for calculation from a context.★
    - **b.** Combine standard function types using arithmetic operations.★

    For example, build a function that models the temperature of a cooling body by adding a constant function to a decaying exponential, and relate these functions to the model.
2. Write arithmetic and geometric sequences both recursively and with an explicit formula them to model situations, and translate between the two forms.★

B. Build new functions from existing functions.

3. Identify the effect on the graph of replacing $f(x)$ by $f(x) + k$, $kf(x)$, $f(kx)$, and $f(x + k)$ for specific values of k (both positive and negative); find the value of k given the graphs. Include linear, quadratic, exponential, and absolute value functions. Utilize technology to experiment with cases and illustrate an explanation of the effects on the graph.
4. Find inverse functions algebraically and graphically.
    - **a.** Solve an equation of the form $f(x) = c$ for a linear function f that has an inverse and write an expression for the inverse.

Linear, Quadratic, and Exponential Models AI.F-LE

A. Construct and compare linear, quadratic, and exponential models and solve problems.

1. Distinguish between situations that can be modeled with linear functions and with exponential functions.★
    - **a.** Prove that linear functions grow by equal differences over equal intervals, and that exponential functions grow by equal factors over equal intervals.★
    - **b.** Recognize situations in which one quantity changes at a constant rate per unit interval relative to another.★
    - **c.** Recognize situations in which a quantity grows or decays by a constant percent rate per unit interval relative to another.★
2. Construct linear and exponential functions, including arithmetic and geometric sequences, given a graph, a description of a relationship, or two input-output pairs (including reading these from a table).★
3. Observe, using graphs and tables, that a quantity increasing exponentially eventually exceeds a quantity increasing linearly, quadratically, or (more generally) as a polynomial function.★

B. Interpret expressions for functions in terms of the situation they model.

5. Interpret the parameters in a linear or exponential function (of the form $f(x) = b^x + k$) in terms of a context.★

Statistics and Probability

Interpreting Categorical and Quantitative Data AI.S-ID

A. Summarize, represent, and interpret data on a single count or measurement variable. Use calculators, spreadsheets, and other technology as appropriate.

1. Represent data with plots on the real number line (dot plots, histograms, and box plots).★
2. Use statistics appropriate to the shape of the data distribution to compare center (median, mean) and spread (interquartile range, standard deviation) of two or more different data sets.★
3. Interpret differences in shape, center, and spread in the context of the data sets, accounting for possible effects of extreme data points (outliers).★

B. Summarize, represent, and interpret data on two categorical and quantitative variables.

5. Summarize categorical data for two categories in two-way frequency tables. Interpret relative frequencies in the context of the data (including joint, marginal, and conditional relative frequencies). Recognize possible associations and trends in the data.★

6. Represent data on two quantitative variables on a scatter plot, and describe how the variables are related.★

- **a.** Fit a linear function to the data and use the fitted function to solve problems in the context of the data. Use functions fitted to data or choose a function suggested by the context (emphasize linear and exponential models).
- **b.** Informally assess the fit of a function by plotting and analyzing residuals.★
- **c.** Fit a linear function for a scatter plot that suggests a linear association.★

C. Interpret linear models.

7. Interpret the slope (rate of change) and the intercept (constant term) of a linear model in the context of the data.★

8. Compute (using technology) and interpret the correlation coefficient of a linear fit.★

9. Distinguish between correlation and causation.

Model Traditional Pathway: Model Geometry [GEO]

Introduction

The fundamental purpose of the Model Geometry course is to formalize and extend students' geometric experiences from the middle grades. In this high school Model Geometry course, students explore more complex geometric situations and deepen their explanations of geometric relationships by presenting and hearing formal mathematical arguments. Important differences exist between this course and the historical approach taken in geometry classes. For example, transformations are emphasized in this course. Close attention should be paid to the introductory content for the Geometry conceptual category.

For the high school Model Geometry course, instructional time should focus on six critical areas: (1) establish criteria for congruence of triangles based on rigid motions; (2) establish criteria for similarity of triangles based on dilations and proportional reasoning; (3) informally develop explanations of circumference, area, and volume formulas; (4) apply the Pythagorean Theorem to the coordinate plane; (5) prove basic geometric theorems; and (6) extend work with probability.

1. Students have prior experience with drawing triangles based on given measurements and performing rigid motions including translations, reflections, and rotations. They have used these to develop notions about what it means for two objects to be congruent. In this course, students establish triangle congruence criteria, based on analyses of rigid motions and formal constructions. They use triangle congruence as a familiar foundation for the development of formal proof. Students prove theorems—using a variety of formats including deductive and inductive reasoning and proof by contradiction—and solve problems about triangles, quadrilaterals, and other polygons. They apply reasoning to complete geometric constructions and explain why they work.
2. Students apply their earlier experience with dilations and proportional reasoning to build a formal understanding of similarity. They identify criteria for similarity of triangles, use similarity to solve problems, and apply similarity in right triangles to understand right triangle trigonometry, with particular attention to special right triangles and the Pythagorean Theorem. Students derive the Laws of Sines and Cosines in order to find missing measures of general (not necessarily right) triangles, building on their work with quadratic equations done in Model Algebra I. They are able to distinguish whether three given measures (angles or sides) define 0, 1, 2, or infinitely many triangles.
3. Students' experience with three-dimensional objects is extended to include informal explanations of circumference, area, and volume formulas. Additionally, students apply their knowledge of two-dimensional shapes to consider the shapes of cross-sections and the result of rotating a two-dimensional object about a line.
4. Building on their work with the Pythagorean Theorem in eighth grade to find distances, students use the rectangular coordinate system to verify geometric relationships, including properties of special triangles and quadrilaterals, and slopes of parallel and perpendicular lines, which relates back to work done in the Model Algebra I course. Students continue their study of quadratics by connecting the geometric and algebraic definitions of the parabola.
5. Students prove basic theorems about circles, with particular attention to perpendicularity and inscribed angles, in order to see symmetry in circles and as an application of triangle congruence criteria. They study relationships among segments on chords, secants, and tangents as an application of similarity. In the Cartesian coordinate system, students use the distance formula to write the equation of a circle when given the radius and the coordinates of its center. Given an equation of a circle, they draw the graph in the coordinate plane, and apply techniques for solving quadratic equations—which relates back to work done in the Model Algebra I course—to determine intersections between lines and circles or parabolas and between two circles.
6. Building on probability concepts that began in the middle grades, students use the language of set theory to expand their ability to compute and interpret theoretical and experimental probabilities for

compound events, attending to mutually exclusive events, independent events, and conditional probability. Students should make use of geometric probability models wherever possible. They use probability to make informed decisions.

The Standards for Mathematical Practice complement the content standards so that students increasingly engage with the subject matter as they grow in mathematical maturity and expertise throughout the elementary, middle, and high school years.

Model Traditional Pathway: Model Geometry Overview [GEO]

Number and Quantity

Quantities

- **A.** Reason quantitatively and use units to solve problems.

Geometry

Congruence

- **A.** Experiment with transformations in the plane.
- **B.** Understand congruence in terms of rigid motions.
- **C.** Prove geometric theorems and, when appropriate, the converse of theorems.
- **D.** Make geometric constructions.

Similarity, Right Triangles, and Trigonometry

- **A.** Understand similarity in terms of transformations.
- **B.** Prove theorems involving similarity.
- **C.** Define trigonometric ratios and solve problems involving right triangles.
- **D.** Apply trigonometry to general triangles.

Circles

- **A.** Understand and apply theorems about circles.
- **B.** Find arc lengths and area of sectors of circles.

Expressing Geometric Properties with Equations

- **A.** Translate between the geometric description and the equation for a conic section.
- **B.** Use coordinates to prove simple geometric theorems algebraically.

Geometric Measurement and Dimension

- **A.** Explain volume formulas and use them to solve problems.
- **B.** Visualize relationships between two-dimensional and three-dimensional objects.

Modeling with Geometry

- **A.** Apply geometric concepts in modeling situations.

Standards for Mathematical Practice

1. Make sense of problems and persevere in solving them.
2. Reason abstractly and quantitatively.
3. Construct viable arguments and critique the reasoning of others.
4. Model with mathematics.
5. Use appropriate tools strategically.
6. Attend to precision.
7. Look for and make use of structure.
8. Look for and express regularity in repeated reasoning.

Statistics and Probability

Conditional Probability and the Rules of Probability

- **A.** Understand independence and conditional probability and use them to interpret data from simulations or experiments.
- **B.** Use the rules of probability to compute probabilities of compound events in a uniform probability model.

Model Traditional Pathway: Model Geometry Content Standards [GEO]

Number and Quantity

Quantities GEO.N-Q

A. Reason quantitatively and use units to solve problems.

3. Choose a level of accuracy appropriate to limitations on measurement when reporting quantities.★
    a. Describe the effects of approximate error in measurement and rounding on measurements and on computed values from measurements. Identify significant figures in recorded measures and computed values based on the context given and the precision of the tools used to measure.★

Geometry

Congruence GEO.G-CO

A. Experiment with transformations in the plane.

1. Know precise definitions of angle, circle, perpendicular line, parallel line, and line segment, based on the undefined notions of point, line, distance along a line, and distance around a circular arc.
2. Represent transformations in the plane using, e.g., transparencies and geometry software; describe transformations as functions that take points in the plane as inputs and give other points as outputs. Compare transformations that preserve distance and angle to those that do not (e.g., translation versus horizontal stretch).
3. Given a rectangle, parallelogram, trapezoid, or regular polygon, describe the rotations and reflections that carry it onto itself.
4. Develop definitions of rotations, reflections, and translations in terms of angles, circles, perpendicular lines, parallel lines, and line segments.
5. Given a geometric figure and a rotation, reflection, or translation, draw the transformed figure using graph paper, tracing paper, or geometry software. Specify a sequence of transformations that will carry a given figure onto another.

B. Understand congruence in terms of rigid motions.

6. Use geometric descriptions of rigid motions to transform figures and to predict the effect of a given rigid motion on a given figure; given two figures, use the definition of congruence in terms of rigid motions to decide if they are congruent.
7. Use the definition of congruence in terms of rigid motions to show that two triangles are congruent if and only if corresponding pairs of sides and corresponding pairs of angles are congruent.
8. Explain how the criteria for triangle congruence (ASA, SAS, and SSS) follow from the definition of congruence in terms of rigid motions.

C. Prove geometric theorems and, when appropriate, the converse of theorems.

9. Prove theorems about lines and angles. Theorems include: vertical angles are congruent; when a transversal crosses parallel lines, alternate interior angles are congruent and corresponding angles are congruent, and conversely prove lines are parallel; points on a perpendicular bisector of a line segment are exactly those equidistant from the segment's endpoints.
10. Prove theorems about triangles. Theorems include: measures of interior angles of a triangle sum to 180°; base angles of isosceles triangles are congruent, and conversely prove a triangle is isosceles; the segment joining midpoints of two sides of a triangle is parallel to the third side and half the length; the medians of a triangle meet at a point.
11. Prove theorems about parallelograms. Theorems include: opposite sides are congruent, opposite angles are congruent, the diagonals of a parallelogram bisect each other, and conversely, rectangles are parallelograms with congruent diagonals.
    a. Prove theorems about polygons. Theorems include the measures of interior and exterior angles. Apply properties of polygons to the solutions of mathematical and contextual problems.

D. Make geometric constructions.

12. Make formal geometric constructions with a variety of tools and methods (compass and straightedge, string, reflective devices, paper folding, dynamic geometric software, etc.). Constructions include: copying a segment; copying an angle; bisecting a segment; bisecting an angle; constructing perpendicular lines, including the perpendicular bisector of a line segment; and constructing a line parallel to a given line through a point not on the line.
13. Construct an equilateral triangle, a square, and a regular hexagon inscribed in a circle.

Similarity, Right Triangles, and Trigonometry GEO.G-SRT

A. Understand similarity in terms of similarity transformations.

1. Verify experimentally the properties of dilations given by a center and a scale factor:
    - **a.** A dilation takes a line not passing through the center of the dilation to a parallel line, and leaves a line passing through the center unchanged.
    - **b.** The dilation of a line segment is longer or shorter in the ratio given by the scale factor.
2. Given two figures, use the definition of similarity in terms of similarity transformations to decide if they are similar; explain using similarity transformations the meaning of similarity for triangles as the equality of all corresponding pairs of angles and the proportionality of all corresponding pairs of sides.
3. Use the properties of similarity transformations to establish the Angle-Angle (AA) criterion for two triangles to be similar.

B. Prove theorems involving similarity.

4. Prove theorems about triangles. Theorems include: a line parallel to one side of a triangle divides the other two proportionally, and conversely; the Pythagorean Theorem proved using triangle similarity.
5. Use congruence and similarity criteria for triangles to solve problems and to prove relationships in geometric figures.

C. Define trigonometric ratios and solve problems involving right triangles.

6. Understand that by similarity, side ratios in right triangles are properties of the angles in the triangle, leading to definitions of trigonometric ratios for acute angles.
7. Explain and use the relationship between the sine and cosine of complementary angles.
8. Use trigonometric ratios and the Pythagorean Theorem to solve right triangles in applied problems.★

D. Apply trigonometry to general triangles.

9. (+) Derive the formula $A = ½ ab \sin(C)$ for the area of a triangle by drawing an auxiliary line from a vertex perpendicular to the opposite side.
10. (+) Prove the Laws of Sines and Cosines and use them to solve problems.
11. (+) Understand and apply the Law of Sines and the Law of Cosines to find unknown measurements in right and non-right triangles (e.g., surveying problems, resultant forces).

Circles GEO.G-C

A. Understand and apply theorems about circles.

1. Prove that all circles are similar.
2. Identify and describe relationships among inscribed angles, radii, and chords. Include the relationship between central, inscribed, and circumscribed angles; inscribed angles on a diameter are right angles; the radius of a circle is perpendicular to the tangent where the radius intersects the circle.
3. Construct the inscribed and circumscribed circles of a triangle, and prove properties of angles for a quadrilateral and other polygons inscribed in a circle.
4. (+) Construct a tangent line from a point outside a given circle to the circle.

B. Find arc lengths and areas of sectors of circles.

5. Derive, using similarity, the fact that the length of the arc intercepted by an angle is proportional to the radius, and define the radian measure of the angle as the constant of proportionality; derive the formula for the area of a sector.

Expressing Geometric Properties with Equations GEO.G-GPE

A. Translate between the geometric description and the equation for a conic section.

1. Derive the equation of a circle of given center and radius using the Pythagorean Theorem; complete the square to find the center and radius of a circle given by an equation.
2. Derive the equation of a parabola given a focus and directrix.

B. Use coordinates to prove simple geometric theorems algebraically.

4. Use coordinates to prove simple geometric theorems algebraically, including the distance formula and its relationship to the Pythagorean Theorem.

   For example, prove or disprove that a figure defined by four given points in the coordinate plane is a rectangle; prove or disprove that the point (1,$\sqrt{3}$) lies on the circle centered at the origin and containing the point (0, 2).

5. Prove the slope criteria for parallel and perpendicular lines and use them to solve geometric problems (e.g., find the equation of a line parallel or perpendicular to a given line that passes through a given point).
6. Find the point on a directed line segment between two given points that partitions the segment in a given ratio.
7. Use coordinates to compute perimeters of polygons and areas of triangles and rectangles (e.g., using the distance formula).★

Geometric Measurement and Dimension GEO.G-GMD

A. Explain volume formulas and use them to solve problems.

1. Give an informal argument for the formulas for the circumference of a circle, area of a circle, volume of a cylinder, pyramid, and cone. Use dissection arguments, Cavalieri's principle, and informal limit arguments.
2. (+) Give an informal argument using Cavalieri's principle for the formulas for the volume of a sphere and other solid figures.
3. Use volume formulas for cylinders, pyramids, cones, and spheres to solve problems.★

B. Visualize relationships between two-dimensional and three-dimensional objects.

4. Identify the shapes of two-dimensional cross-sections of three-dimensional objects, and identify three-dimensional objects generated by rotations of two-dimensional objects.

Modeling with Geometry GEO.G-MG

A. Apply geometric concepts in modeling situations.

1. Use geometric shapes, their measures, and their properties to describe objects (e.g., modeling a tree trunk or a human torso as a cylinder).★
2. Apply concepts of density based on area and volume in modeling situations (e.g., persons per square mile, BTUs per cubic foot).★
3. Apply geometric methods to solve design problems (e.g., designing an object or structure to satisfy physical constraints or minimize cost; working with typographic grid systems based on ratios).★
4. Use dimensional analysis for unit conversions to confirm that expressions and equations make sense.★

Statistics and Probability

Conditional Probability and the Rules of Probability GEO.S-CP

A. Understand independence and conditional probability and use them to interpret data from simulations or experiments.

1. Describe events as subsets of a sample space (the set of outcomes) using characteristics (or categories) of the outcomes, or as unions, intersections, or complements of other events ("or," "and," "not").★
2. Understand that two events *A* and *B* are independent if the probability of *A* and *B* occurring together is the product of their probabilities, and use this characterization to determine if they are independent.★
3. Understand the conditional probability of *A* given *B* as $^{P(A \text{ and } B)}/_{P(B)}$, and interpret independence of *A* and *B* as saying that the conditional probability of A given B is the same as the probability of *A*, and the conditional probability of *B* given *A* is the same as the probability of *B*.★

4. Construct and interpret two-way frequency tables of data when two categories are associated with each object being classified. Use the two-way table as a sample space to decide if events are independent and to approximate conditional probabilities.★

   For example, collect data from a random sample of students in your school on their favorite subject among math, science, and English. Estimate the probability that a randomly selected student from your school will favor science given that the student is in tenth grade. Do the same for other subjects and compare the results.

5. Recognize and explain the concepts of conditional probability and independence in everyday language and everyday situations.★

   For example, compare the chance of having lung cancer if you are a smoker with the chance of being a smoker if you have lung cancer.

B. Use the rules of probability to compute probabilities of compound events in a uniform probability model.

6. Find the conditional probability of A given B as the fraction of B's outcomes that also belong to A, and interpret the answer in terms of the model.★
7. Apply the Addition Rule, $P(A \text{ or } B) = P(A) + P(B) - P(A \text{ and } B)$, and interpret the answer in terms of the model.★
8. (+) Apply the general Multiplication Rule in a uniform probability model, $P(A \text{ and } B) = P(A)P(B|A) = P(B)P(A|B)$, and interpret the answer in terms of the model.★
9. (+) Use permutations and combinations to compute probabilities of compound events and solve problems.★

Model Traditional Pathway: Model Algebra II [AII]

Introduction

Building on their work with linear, quadratic, and exponential functions, students extend their repertoire of functions to include logarithmic, polynomial, rational, and radical functions in the Model Algebra II course. Students work closely with the expressions that define the functions, are facile with algebraic manipulations of expressions, and continue to expand and hone their abilities to model situations and to solve equations, including solving quadratic equations over the set of complex numbers and solving exponential equations using the properties of logarithms.

For the high school Model Algebra II course, instructional time should focus on four critical areas: (1) relate arithmetic of rational expressions to arithmetic of rational numbers; (2) expand understandings of functions and graphing to include trigonometric functions; (3) synthesize and generalize functions and extend understanding of exponential functions to logarithmic functions; and (4) relate data display and summary statistics to probability and explore a variety of data collection methods.

1. A central theme of this Model Algebra II course is that the arithmetic of rational expressions is governed by the same rules as the arithmetic of rational numbers. Students explore the structural similarities between the system of polynomials and the system of integers. They draw on analogies between polynomial arithmetic and base-ten computation, focusing on properties of operations, particularly the distributive property. Connections are made between multiplication of polynomials with multiplication of multi-digit integers, and division of polynomials with long division of integers. Students identify zeros of polynomials, including complex zeros of quadratic polynomials, and make connections between zeros of polynomials and solutions of polynomial equations. The Fundamental Theorem of Algebra is examined.
2. Building on their previous work with functions and on their work with trigonometric ratios and circles in the Model Geometry course, students now use the coordinate plane to extend trigonometry to model periodic phenomena.
3. Students synthesize and generalize what they have learned about a variety of function families. They extend their work with exponential functions to include solving exponential equations with logarithms. They explore the effects of transformations on graphs of diverse functions, including functions arising in an application, in order to abstract the general principle that transformations on a graph always have the same effect regardless of the type of the underlying function. They identify appropriate types of functions to model a situation, they adjust parameters to improve the model, and they compare models by analyzing appropriateness of fit and making judgments about the domain over which a model is a good fit. The description of modeling as "*the process of choosing and using mathematics and statistics to analyze empirical situations, to understand them better, and to make decisions*" is at the heart of this Model Algebra II course. The narrative discussion and diagram of the modeling cycle should be considered when knowledge of functions, statistics, and geometry is applied in a modeling context.
4. Students see how the visual displays and summary statistics they learned in earlier grades relate to different types of data and to probability distributions. They identify different ways of collecting data—including sample surveys, experiments, and simulations—and the role that randomness and careful design play in the conclusions that can be drawn.

The Standards for Mathematical Practice complement the content standards so that students increasingly engage with the subject matter as they grow in mathematical maturity and expertise throughout the elementary, middle, and high school years.

Model Traditional Pathway: Model Algebra II Overview [AII]

Number and Quantity

The Complex Number System

- **A.** Perform arithmetic operations with complex numbers.
- **C.** Use complex numbers in polynomial identities and equations.

Vector and Matrix Quantities

- **A.** Represent and model with vector quantities.
- **C.** Perform operations on matrices and use matrices in applications.

Algebra

Seeing Structure in Expressions

- **A.** Interpret the structure of exponential, polynomial, and rational expressions.
- **B.** Write expressions in equivalent forms to solve problems.

Arithmetic with Polynomials and Rational Expressions

- **A.** Perform arithmetic operations on polynomials.
- **B.** Understand the relationship between zeros and factors of polynomials.
- **C.** Use polynomial identities to solve problems.
- **D.** Rewrite rational expressions.

Creating Equations

- **A.** Create equations that describe numbers or relationships.

Reasoning with Equations and Inequalities

- **A.** Understand solving equations as a process of reasoning and explain the reasoning.
- **D.** Represent and solve equations and inequalities graphically.

Standards for Mathematical Practice

1. Make sense of problems and persevere in solving them.
2. Reason abstractly and quantitatively.
3. Construct viable arguments and critique the reasoning of others.
4. Model with mathematics.
5. Use appropriate tools strategically.
6. Attend to precision.
7. Look for and make use of structure.
8. Look for and express regularity in repeated reasoning.

Functions

Interpreting Functions

- **B.** Interpret functions that arise in applications in terms of the context (polynomial, rational, square root and cube root, trigonometric, and logarithmic functions).
- **C.** Analyze functions using different representations.

Building Functions

- **A.** Build a function that models a relationship between two quantities.
- **B.** Build new functions from existing functions.

Linear, Quadratic, and Exponential Models

- **A.** Construct and compare linear, quadratic, and exponential models and solve problems.

Trigonometric Functions

- **A.** Extend the domain of trigonometric functions using the unit circle.
- **B.** Model periodic phenomena with trigonometric functions.
- **C.** Prove and apply trigonometric identities.

Statistics and Probability

Interpreting Categorical and Quantitative Data

A. Summarize, represent and interpret data on a single count or measurement variable. Use calculators, spreadsheets, and other technology as appropriate.

Making Inferences and Justifying Conclusions

A. Understand and evaluate random processes underlying statistical experiments.

B. Make inferences and justify conclusions from sample surveys, experiments, and observational studies.

Using Probability to Make Decisions

B. Use probability to evaluate outcomes of decisions.

Model Traditional Pathway: Model Algebra II Content Standards [AII]

Number and Quantity

The Complex Number System AII.N-CN

A. Perform arithmetic operations with complex numbers.

1. Know there is a complex number i such that $i^2 = -1$, and every complex number has the form $a + bi$ with x-a and b real.
2. Use the relation $i^2 = -1$ and the Commutative, Associative, and Distributive properties to add, subtract, and multiply complex numbers.

C. Use complex numbers in polynomial identities and equations.

7. Solve quadratic equations with real coefficients that have complex solutions.
8. (+) Extend polynomial identities to the complex numbers.

   For example, rewrite $x^2 + 4$ as $(x + 2i)(x - 2i)$.
9. (+) Know the Fundamental Theorem of Algebra; show that it is true for quadratic polynomials.

Vector and Matrix Quantities AII.N-VM

A. Represent and model with vector quantities.

1. (+) Recognize vector quantities as having both magnitude and direction. Represent vector quantities by directed line segments, and use appropriate symbols for vectors and their magnitudes (e.g., $\mathbf{v}$, $|\mathbf{v}|$, $||\mathbf{v}||$, v).
3. (+) Solve problems involving velocity and other quantities that can be represented by vectors.

C. Perform operations on matrices and use matrices in applications.

6. (+) Use matrices to represent and manipulate data, e.g., to represent payoffs or incidence relationships in a network.
8. (+) Add, subtract, and multiply matrices of appropriate dimensions.
12. (+) Work with 2 × 2 matrices as transformations of the plane, and interpret the absolute value of the determinant in terms of area.

Algebra

Seeing Structure in Expressions AII.A-SSE

A. Interpret the structure of exponential, polynomial, and rational expressions.

1. Interpret expressions that represent a quantity in terms of its context.★
   - **a.** Interpret parts of an expression, such as terms, factors, and coefficients.
   - **b.** Interpret complicated expressions by viewing one or more of their parts as a single entity.

   For example, interpret $P(1 + r)^n$ as the product of P and a factor not depending on P.
2. Use the structure of an expression to identify ways to rewrite it.

   For example, see $x^4 - y^4$ as $(x^2)^2 - (y^2)^2$, thus recognizing it as a difference of squares that can be factored as $(x^2 - y^2)(x^2 + y^2)$and further factored $(x-y)(x+y)(x-yi)(x+yi)$.

B. Write expressions in equivalent forms to solve problems.

4. Derive the formula for the sum of a finite geometric series (when the common ratio is not 1), and use the formula to solve problems.★

   For example, calculate mortgage payments.

Arithmetic with Polynomials and Rational Expressions AII.A-APR

A. Perform arithmetic operations on polynomials.

1. Understand that polynomials form a system analogous to the integers, namely, they are closed under certain operations.

   a. Perform operations on polynomial expressions (addition, subtraction, multiplication, and division), and compare the system of polynomials to the system of integers when performing operations.

B. Understand the relationship between zeros and factors of polynomials.

2. Know and apply the Remainder Theorem: For a polynomial $p(x)$ and a number a, the remainder on division by $x – a$ is $p(a)$, so $p(a) = 0$ if and only if $(x – a)$ is a factor of $p(x)$.

3. Identify zeros of polynomials when suitable factorizations are available, and use the zeros to construct a rough graph of the function defined by the polynomial.

C. Use polynomial identities to solve problems.

4. Prove polynomial identities and use them to describe numerical relationships.

For example, the polynomial identity $(x^2 + y^2)^2 = (x^2 – y^2)^2 + (2xy)^2$ can be used to generate Pythagorean triples.

5. (+) Know and apply the Binomial Theorem for the expansion of $(x + y)^n$ in powers of x and y for a positive integer n, where x and y are any numbers, with coefficients determined for example by Pascal's Triangle.

D. Rewrite rational expressions.

6. Rewrite simple rational expressions in different forms; write $^{a(x)}/_{b(x)}$ in the form $q(x) + {}^{r(x)}/_{b(x),}$ where $a(x)$, $b(x)$, $q(x)$, and $r(x)$ are polynomials with the degree of $r(x)$ less than the degree of $b(x)$, using inspection, long division, or, for the more complicated examples, a computer algebra system.

7. (+) Understand that rational expressions form a system analogous to the rational numbers, closed under addition, subtraction, multiplication, and division by a nonzero rational expression; add, subtract, multiply, and divide rational expressions.

Creating Equations AII.A-CED

A. Create equations that describe numbers or relationships.

1. Create equations and inequalities in one variable and use them to solve problems. Include equations arising from simple root and rational functions and exponential functions.★

2. Create equations in two or more variables to represent relationships between quantities; graph equations on coordinate axes with labels and scales.★

3. Represent constraints by equations or inequalities, and by systems of equations and/or inequalities, and interpret solutions as viable or non-viable options in a modeling context.

For example, represent equations describing satellites orbiting Earth and constraints on Earth's size and atmosphere.

Reasoning with Equations and Inequalities AII.A-REI

A. Understand solving equations as a process of reasoning and explain the reasoning.

2. Solve simple rational and radical equations in one variable, and give examples showing how extraneous solutions may arise.

D. Represent and solve equations and inequalities graphically.

11. Explain why the x-coordinates of the points where the graphs of the equations $y = f(x)$ and $y = g(x)$ intersect are the solutions of the equation $f(x) = g(x)$; find the solutions approximately, e.g., using technology to graph the functions, make tables of values, or find successive approximations. Include cases where $f(x)$ and/or $g(x)$ are polynomial, rational, and logarithmic functions.★

Functions

Interpreting Functions AII.F-IF

B. Interpret functions that arise in applications in terms of the context (polynomial, rational, square root and cube root, trigonometric, and logarithmic functions).

4. For a function that models a relationship between two quantities, interpret key features of graphs and tables in terms of the quantities, and sketch graphs showing key features given a verbal description of the relationship. Key features include: intercepts; intervals where the function is increasing, decreasing, positive, or negative; relative maximums and minimums; symmetries; end behavior; and periodicity.★

5. Relate the domain of a function to its graph and, where applicable, to the quantitative relationship it describes.★

   For example, if the function h(n) gives the number of person-hours it takes to assemble n engines in a factory, then the positive integers would be an appropriate domain for the function.

6. Calculate and interpret the average rate of change of a function (presented symbolically or as a table) over a specified interval. Estimate the rate of change from a graph.★

C. Analyze functions using different representations.

7. Graph functions expressed symbolically and show key features of the graph; by hand in simple cases and using technology for more complicated cases.★
   - **b.** Graph square root and cube root functions.★
   - **c.** Graph polynomial functions, identifying zeros when suitable factorizations are available, and showing end behavior.★
   - **e.** Graph exponential and logarithmic functions, showing intercepts and end behavior; and trigonometric functions, showing period, midline, and amplitude.★
8. Write a function defined by an expression in different but equivalent forms to reveal and explain different properties of the function.
   - **a.** Use the process of factoring in a polynomial function to show zeros, extreme values, and symmetry of the graph and interpret these in terms of a context.
9. Translate among different representations of functions (algebraically, graphically, numerically in tables, or by verbal descriptions). Compare properties of two functions each represented in a different way.

   For example, given a graph of one polynomial function and an algebraic expression for another, say which has the larger relative maximum and/or smaller relative minimum.

10. Given algebraic, numeric and/or graphical representations of functions, recognize the function as polynomial, rational, logarithmic, exponential, or trigonometric.

Building Functions AII.F-BF

A. Build a function that models a relationship between two quantities.

1. Write a function (simple rational, radical, logarithmic, and trigonometric functions) that describes a relationship between two quantities.★
   - **b.** Combine standard function types using arithmetic operations.★

   For example, build a function that models the temperature of a cooling body by adding a constant function to a decaying exponential, and relate these functions to the model.

B. Build new functions from existing functions.

3. Identify the effect on the graph of replacing $f(x)$ by $f(x) + k$, $k f(x)$, $f(kx)$, and $f(x + k)$ for specific values of k (both positive and negative); find the value of k given the graphs. Include simple rational, radical, logarithmic, and trigonometric functions. Utilize technology to experiment with cases and illustrate an explanation of the effects on the graph using technology. Include recognizing even and odd functions from their graphs and algebraic expressions for them.
4. Find inverse functions algebraically and graphically.
   - **a.** Solve an equation of the form $f(x) = c$ for a simple function f that has an inverse and write an expression for the inverse.

   For example, $f(x) = 2x^3$ or $f(x) = {}^{(x+1)}/_{(x-1)}$ for $x \neq 1$.

Linear, Quadratic, and Exponential Models AII.F-LE

A. Construct and compare linear, quadratic, and exponential models and solve problems.

4. For exponential models, express as a logarithm the solution to $ab^{ct} = d$ where a, c, and d are numbers and the base b is 2, 10, or e; evaluate the logarithm using technology.★

Trigonometric Functions AII.F-TF

A. Extend the domain of trigonometric functions using the unit circle.

1. Understand radian measure of an angle as the length of the arc on the unit circle subtended by the angle.
2. Explain how the unit circle in the coordinate plane enables the extension of trigonometric functions to all real numbers, interpreted as radian measures of angles traversed counterclockwise around the unit circle.

B. Model periodic phenomena with trigonometric functions.

5. Choose trigonometric functions to model periodic phenomena with specified amplitude, frequency, and midline.★

C. Prove and apply trigonometric identities.

8. Prove the Pythagorean identity $\sin^2(\theta) + \cos^2(\theta) = 1$ and use it to find sin(θ), cos(θ), or tan(θ) given sin(θ), cos(θ), or tan(θ) and the quadrant.

Statistics and Probability

Interpreting Categorical and Quantitative Data AII.S-ID

A. Summarize, represent, and interpret data on a single count or measurement variable. Use calculators, spreadsheets, and other technology as appropriate.

4. Use the mean and standard deviation of a data set to fit it to a normal distribution and to estimate population percentages. Recognize that there are data sets for which such a procedure is not appropriate. Use calculators, spreadsheets, and tables to estimate areas under the normal curve.★

Making Inferences and Justifying Conclusions AII.S-IC

A. Understand and evaluate random processes underlying statistical experiments.

1. Understand statistics as a process for making inferences to be made about population parameters based on a random sample from that population.★
2. Decide if a specified model is consistent with results from a given data-generating process, e.g., using simulation.

   For example, a model says a spinning coin falls heads up with probability 0.5. Would a result of five tails in a row cause you to question the model?

B. Make inferences and justify conclusions from sample surveys, experiments, and observational studies.

3. Recognize the purposes of and differences among sample surveys, experiments, and observational studies; explain how randomization relates to each.★
4. Use data from a sample survey to estimate a population mean or proportion; develop a margin of error through the use of simulation models for random sampling.★
5. Use data from a randomized experiment to compare two treatments; use simulations to decide if differences between parameters are significant.★
6. Evaluate reports based on data.★

Using Probability to Make Decisions AII.S-MD

B. Use probability to evaluate outcomes of decisions.

6. (+) Use probabilities to make fair decisions (e.g., drawing by lots, using a random number generator).★
7. (+) Analyze decisions and strategies using probability concepts (e.g., product testing, medical testing, pulling a hockey goalie at the end of a game and replacing the goalie with an extra skater).★

Model Integrated Pathway: Model Mathematics I [MI]

Introduction

The fundamental purpose of the Model Mathematics I course is to formalize and extend the mathematics that students learned in the middle grades.

For the high school Model Mathematics I course, instructional time should focus on six critical areas, each of which is described in more detail below: (1) extend understanding of numerical manipulation to algebraic manipulation; (2) synthesize understanding of function; (3) deepen and extend understanding of linear relationships; (4) apply linear models to data that exhibit a linear trend; (5) establish criteria for congruence based on rigid motions; and (6) apply the Pythagorean Theorem to the coordinate plane.

1. By the end of eighth grade students have had a variety of experiences working with expressions and creating equations. Students become facile with algebraic manipulation in much the same way that they are facile with numerical manipulation. Algebraic facility includes rearranging and collecting terms, factoring, identifying and canceling common factors in rational expressions, and applying properties of exponents. Students continue this work by using quantities to model and analyze situations, to interpret expressions, and to create equations to describe situations.
2. In earlier grades, students define, evaluate, and compare functions, and use them to model relationships among quantities. Students will learn function notation and develop the concepts of domain and range. They move beyond viewing functions as processes that take inputs and yield outputs and start viewing functions as objects in their own right. They explore many examples of functions, including sequences; interpret functions given graphically, numerically, symbolically, and verbally; translate between representations; and understand the limitations of various representations. They work with functions given by graphs and tables, keeping in mind that, depending upon the context, these representations are likely to be approximate and incomplete. Their work includes functions that can be described or approximated by formulas as well as those that cannot. When functions describe relationships between quantities arising from a context, students reason with the units in which those quantities are measured. Students build on and informally extend their understanding of integer exponents to consider exponential functions. They compare and contrast linear and exponential functions, distinguishing between additive and multiplicative change. They interpret arithmetic sequences as linear functions and geometric sequences as exponential functions.
3. By the end of eighth grade, students have learned to solve linear equations in one variable and have applied graphical and algebraic methods to analyze and solve systems of linear equations in two variables. Building on these earlier experiences, students analyze and explain the process of solving an equation, and justify the process used in solving a system of equations. Students develop fluency writing, interpreting, and translating among various forms of linear equations and inequalities, and use them to solve problems. They master the solution of linear equations and apply related solution techniques and the laws of exponents to the creation and solution of simple exponential equations. Students explore systems of equations and inequalities, and they find and interpret their solutions. All of this work is grounded on understanding quantities and on relationships among them.
4. Students' prior experiences with data are the basis for the more formal means of assessing how a model fits data. Students use regression techniques to describe approximately linear relationships among quantities. They use graphical representations and knowledge of the context to make judgments about the appropriateness of linear models. With linear models, they look at residuals to analyze the goodness of fit.
5. In previous grades, students were asked to draw triangles based on given measurements. They also have prior experience with rigid motions: translations, reflections, and rotations, and have used these to develop notions about what it means for two objects to be congruent. Students establish triangle

congruence criteria, based on analyses of rigid motions and formal constructions. They solve problems about triangles, quadrilaterals, and other polygons. They apply reasoning to complete geometric constructions and explain why they work.

6. Building on their work with the Pythagorean Theorem in eighth grade to find distances, students use a rectangular coordinate system to verify geometric relationships, including properties of special triangles and quadrilaterals and slopes of parallel and perpendicular lines.

The Standards for Mathematical Practice complement the content standards so that students increasingly engage with the subject matter as they grow in mathematical maturity and expertise throughout the elementary, middle, and high school years.

Model Integrated Pathway: Model Mathematics I Overview [MI]

Number and Quantity

Quantities

- **A.** Reason quantitatively and use units to solve problems.

Algebra

Seeing Structure in Expressions

- **A.** Interpret the structure of linear and exponential expressions with integer exponents.

Creating Equations

- **A.** Create equations that describe numbers or relationships.

Reasoning with Equations and Inequalities

- **A.** Understand solving equations as a process of reasoning and explain the reasoning.
- **B.** Solve equations and inequalities in one variable.
- **C.** Solve systems of equations.
- **D.** Represent and solve equations and inequalities graphically.

Standards for Mathematical Practice

1. Make sense of problems and persevere in solving them.
2. Reason abstractly and quantitatively.
3. Construct viable arguments and critique the reasoning of others.
4. Model with mathematics.
5. Use appropriate tools strategically.
6. Attend to precision.
7. Look for and make use of structure.
8. Look for and express regularity in repeated reasoning.

Functions

Interpreting Functions

- **A.** Understand the concept of a function and use function notation.
- **B.** Interpret linear and exponential functions having integer exponents that arise in applications in terms of the context.
- **C.** Analyze functions using different representations.

Building Functions

- **A.** Build a function that models a relationship between two quantities.
- **B.** Build new functions from existing functions.

Linear, Quadratic, and Exponential Models

- **A.** Construct and compare linear and exponential models and solve problems.
- **B.** Interpret expressions for functions in terms of the situation they model.

Geometry

Congruence

- **A.** Experiment with transformations in the plane.
- **B.** Understand congruence in terms of rigid motions.
- **D.** Make geometric constructions.

Expressing Geometric Properties with Equations

- **B.** Use coordinates to prove simple geometric theorems algebraically.

Statistics and Probability

Interpreting Categorical and Quantitative Data

A. Summarize, represent, and interpret data on a single count or measurement variable. Use calculators, spreadsheets, and other technology as appropriate.

B. Summarize, represent, and interpret data on two categorical and quantitative variables.

C. Interpret linear models.

Model Integrated Pathway: Model Mathematics I Content Standards [MI]

Number and Quantity

Quantities MI.N-Q

A. Reason quantitatively and use units to solve problems.

1. Use units as a way to understand problems; and to guide the solution of multi-step problems; choose and interpret units consistently in formulas; choose and interpret the scale and the origin in graphs and data displays.★
2. Define appropriate quantities for the purpose of descriptive modeling.★
3. Choose a level of accuracy appropriate to limitations on measurement when reporting quantities.★

Algebra

Seeing Structure in Expressions MI. A-SSE

A. Interpret the structure of linear and exponential expressions with integer exponents.

1. Interpret expressions that represent a quantity in terms of its context.★
    - **a.** Interpret parts of an expression, such as terms, factors, and coefficients.
    - **b.** Interpret complicated expressions by viewing one or more of their parts as a single entity.

For example, interpret P(1 + r)n as the product of P and a factor not depending on P.

Creating Equations MI. A-CED

A. Create equations that describe numbers or relationships.

1. Create equations and inequalities in one variable and use them to solve problems. Include equations arising from linear and exponential functions with integer exponents.★
2. Create equations in two or more variables to represent relationships between quantities; graph equations on coordinate axes with labels and scales.★
3. Represent constraints by linear equations or inequalities, and by systems of linear equations and/or inequalities, and interpret solutions as viable or non-viable options in a modeling context.★

For example, represent inequalities describing nutritional and cost constraints on combinations of different foods.

4. Rearrange formulas to highlight a quantity of interest, using the same reasoning (Properties of equality) as in solving equations.★

For example, rearrange Ohm's law, $V = IR$, to solve for resistance, R. Manipulate variables in formulas used in financial contexts such as for simple interest, $I=Prt$.

Reasoning with Equations and Inequalities MI.A-REI

A. Understand solving equations as a process of reasoning and explain the reasoning.

1. Explain each step in solving a simple linear equation as following from the equality of numbers asserted at the previous step, starting from the assumption that the original equation has a solution. Construct a viable argument to justify or refute a solution method.

B. Solve equations and inequalities in one variable.

3. Solve linear equations and inequalities in one variable, including equations with coefficients represented by letters.
    - **a.** Solve linear equations and inequalities in one variable involving absolute value.

C. Solve systems of equations.

5. Prove that, given a system of two equations in two variables, replacing one equation by the sum of that equation and a multiple of the other produces a system with the same solutions.
6. Solve systems of linear equations exactly and approximately (e.g., with graphs), focusing on pairs of linear equations in two variables.

D. Represent and solve equations and inequalities graphically.

10. Understand that the graph of an equation in two variables is the set of all its solutions plotted in the coordinate plane, often forming a curve (which could be a line). Show that any point on the graph of an equation in two variables is a solution to the equation.

11. Explain why the x-coordinates of the points where the graphs of the equations $y = f(x)$ and $y = g(x)$ intersect are the solutions of the equation $f(x) = g(x)$; find the solutions approximately, e.g., using technology to graph the functions and/or make tables of values. Include cases where $f(x)$ and/or $g(x)$ are linear and exponential functions.★

12. Graph the solutions of a linear inequality in two variables as a half-plane (excluding the boundary in the case of a strict inequality), and graph the solution set of a system of linear inequalities in two variables as the intersection of the corresponding half-planes.

Functions

Interpreting Functions MI. F-IF

A. Understand the concept of a function and use function notation.

1. Understand that a function from one set (called the domain) to another set (called the range) assigns to each element of the domain exactly one element of the range. If f is a function and x is an element of its domain, then $f(x)$ denotes the output of f corresponding to the input x. The graph of f is the graph of the equation $y = f(x)$.

2. Use function notation, evaluate functions for inputs in their domains, and interpret statements that use function notation in terms of a context.

For example, given a function representing a car loan, determine the balance of the loan at different points in time.

3. Recognize that sequences are functions, sometimes defined recursively, whose domain is a subset of the integers.

For example, the Fibonacci sequence is defined recursively by $f(0) = f(1) = 1$, $f(n + 1) = f(n) + f(n - 1)$ for $n \geq 1$.

B. Interpret linear and exponential functions having integer exponents that arise in applications in terms of the context.

4. For a function that models a relationship between two quantities, interpret key features of graphs and tables in terms of the quantities, and sketch graphs showing key features given a verbal description of the relationship. Key features include: intercepts; intervals where the function is increasing, decreasing, positive, or negative; relative maximums and minimums; symmetries; and end behavior.★

5. Relate the domain of a function to its graph and, where applicable, to the quantitative relationship it describes.★

For example, if the function h(n) gives the number of person-hours it takes to assemble n engines in a factory, then the positive integers would be an appropriate domain for the function.

6. Calculate and interpret the average rate of change of a function (presented symbolically or as a table) over a specified interval. Estimate the rate of change from a graph.★

C. Analyze functions using different representations.

7. Graph functions expressed symbolically and show key features of the graph, by hand in simple cases and using technology for more complicated cases.★

- **a.** Graph linear functions and show intercepts.★
- **e.** Graph exponential functions, showing intercepts and end behavior.★

9. Translate among different representations of functions: (algebraically, graphically, numerically in tables, or by verbal descriptions). Compare properties of two functions each represented in a different way.

For example, given a graph of one exponential function and an algebraic expression for another, say which has the larger y-intercept.

Building Functions MI.F-BF

A. Build a function that models a relationship between two quantities.

1. Write linear and exponential functions that describe a relationship between two quantities.★

    - **a.** Determine an explicit expression, a recursive process, or steps for calculation from a context.★
    - **b.** Combine standard function types using arithmetic operations.★

    For example, build a function that models the temperature of a cooling body by adding a constant function to a decaying exponential, and relate these functions to the model.

- **2.** Write arithmetic and geometric sequences both recursively and with an explicit formula, use them to model situations, and translate between the two forms.★

B. Build new functions from existing functions.

- **3.** Identify the effect on the graph of replacing $f(x)$ by $f(x) + k$, $kf(x)$, $f(kx)$, and $f(x + k)$ for specific values of k (both positive and negative); find the value of k given the graphs. Include linear and exponential models. (Focus on vertical translations for exponential functions). Utilize technology to experiment with cases and illustrate an explanation of the effects on the graph.

Linear, Quadratic, and Exponential Models MI.F-LE

A. Construct and compare linear and exponential models and solve problems.

- **1.** Distinguish between situations that can be modeled with linear functions and with exponential functions.★
    - **a.** Prove that linear functions grow by equal differences over equal intervals, and that exponential functions grow by equal factors over equal intervals.★
    - **b.** Recognize situations in which one quantity changes at a constant rate per unit interval relative to another.★
    - **c.** Recognize situations in which a quantity grows or decays by a constant percent rate per unit interval relative to another.★
- **2.** Construct linear and exponential functions, including arithmetic and geometric sequences, given a graph, a description of a relationship, or two input-output pairs (including reading these from a table).★
- **3.** Observe using graphs and tables that a quantity increasing exponentially eventually exceeds a quantity increasing linearly.★

B. Interpret expressions for functions in terms of the situation they model.

- **5.** Interpret the parameters in a linear function or exponential function (of the form $f(x) = b^x + k$) in terms of a context.★

Geometry

Congruence MI.G-CO

A. Experiment with transformations in the plane.

- **1.** Know precise definitions of angle, circle, perpendicular line, parallel line, and line segment, based on the undefined notions of point, line, distance along a line, and distance around a circular arc.
- **2.** Represent transformations in the plane using, e.g., transparencies and geometry software; describe transformations as functions that take points in the plane as inputs and give other points as outputs. Compare transformations that preserve distance and angle to those that do not (e.g., translation versus horizontal stretch).
- **3.** Given a rectangle, parallelogram, trapezoid, or regular polygon, describe the rotations and reflections that carry it onto itself.
- **4.** Develop definitions of rotations, reflections, and translations in terms of angles, circles, perpendicular lines, parallel lines, and line segments.
- **5.** Given a geometric figure and a rotation, reflection, or translation, draw the transformed figure using, e.g., graph paper, tracing paper, or geometry software. Specify a sequence of transformations that will carry a given figure onto another.

B. Understand congruence in terms of rigid motions.

- **6.** Use geometric descriptions of rigid motions to transform figures and to predict the effect of a given rigid motion on a given figure; given two figures, use the definition of congruence in terms of rigid motions to decide if they are congruent.
- **7.** Use the definition of congruence in terms of rigid motions to show that two triangles are congruent if and only if corresponding pairs of sides and corresponding pairs of angles are congruent.

8. Explain how the criteria for triangle congruence (ASA, SAS, and SSS) follow from the definition of congruence in terms of rigid motions.

D. Make geometric constructions.

12. Make formal geometric constructions with a variety of tools and methods (compass and straightedge, string, reflective devices, paper folding, dynamic geometric software, etc.). Copying a segment; copying an angle; bisecting a segment; bisecting an angle; constructing perpendicular lines, including the perpendicular bisector of a line segment; and constructing a line parallel to a given line through a point not on the line.
13. Construct an equilateral triangle, a square, and a regular hexagon inscribed in a circle.

Expressing Geometric Properties with Equations MI.G-GPE

B. Use coordinates to prove simple geometric theorems algebraically.

5. Prove the slope criteria for parallel and perpendicular lines and use them to solve geometric problems (e.g., find the equation of a line parallel or perpendicular to a given line that passes through a given point).
7. Use coordinates to compute perimeters of polygons and areas of triangles and rectangles (e.g., using the distance formula).★

Statistics and Probability

Interpreting Categorical and Quantitative Data MI.S-ID

A. Summarize, represent, and interpret data on a single count or measurement variable. Use calculators, spreadsheets, and other technology as appropriate.

1. **Represent data with plots on the real number line (dot plots, histograms, and box plots).★**
2. Use statistics appropriate to the shape of the data distribution to compare center (median, mean) and spread (interquartile range, standard deviation) of two or more different data sets.★
3. Interpret differences in shape, center, and spread in the context of the data sets, accounting for possible effects of extreme data points (outliers).★

B. Summarize, represent, and interpret data on two categorical and quantitative variables.

5. Summarize categorical data for two categories in two-way frequency tables. Interpret relative frequencies in the context of the data (including joint, marginal, and conditional relative frequencies). Recognize possible associations and trends in the data.★
6. Represent data on two quantitative variables on a scatter plot, and describe how the variables are related.★
    a. Fit a linear function to the data and use the fitted function to solve problems in the context of the data. Use given functions fitted to data or choose a function suggested by the context. Emphasize linear and exponential models.★
    b. Informally assess the fit of a function by plotting and analyzing residuals.★
    c. Fit a linear function for a scatter plot that suggests a linear association.★

C. Interpret linear models.

7. Interpret the slope (rate of change) and the intercept (constant term) of a linear model in the context of the data.★
8. Compute (using technology) and interpret the correlation coefficient of a linear fit.★
9. Distinguish between correlation and causation.★

Model Integrated Pathway: Model Mathematics II [MII]

Introduction

The focus of the Model Mathematics II course is on quadratic expressions, equations, and functions; comparing their characteristics and behavior to those of linear and exponential relationships from Model Mathematics I.

For the high school Model Mathematics II course, instructional time should focus on five critical areas: (1) extend the laws of exponents to rational exponents; (2) compare key characteristics of quadratic functions with those of linear and exponential functions; (3) create and solve equations and inequalities involving linear, exponential, and quadratic expressions; (4) extend work with probability; and (5) establish criteria for similarity of triangles based on dilations and proportional reasoning.

1. Students extend the laws of exponents to rational exponents and explore distinctions between rational and irrational numbers by considering their decimal representations. Students learn that when quadratic equations do not have real solutions, the number system must be extended so that solutions exist; analogous to the way in which extending the whole numbers to the negative numbers allows $x + 1 = 0$ to have a solution. Students explore relationships between number systems: whole numbers, integers, rational numbers, real numbers, and complex numbers. The guiding principle is that equations with no solutions in one number system may have solutions in a larger number system.
2. Students consider quadratic functions, comparing the key characteristics of quadratic functions to those of linear and exponential functions. They select from among these functions to model phenomena. Students learn to anticipate the graph of a quadratic function by interpreting various forms of quadratic expressions. In particular, they identify the real solutions of a quadratic equation as the zeros of a related quadratic function. When quadratic equations do not have real solutions, students learn that that the graph of the related quadratic function does not cross the horizontal axis. They expand their experience with functions to include more specialized functions—absolute value, step, and those that are piecewise-defined.
3. Students begin by focusing on the structure of expressions, rewriting expressions to clarify and reveal aspects of the relationship they represent. They create and solve equations, inequalities, and systems of equations involving exponential and quadratic expressions.
4. Building on probability concepts that began in the middle grades, students use the language of set theory to expand their ability to compute and interpret theoretical and experimental probabilities for compound events, attending to mutually exclusive events, independent events, and conditional probability. Students should make use of geometric probability models wherever possible. They use probability to make informed decisions.
5. Students apply their earlier experience with dilations and proportional reasoning to build a formal understanding of similarity. They identify criteria for similarity of triangles, use similarity to solve problems, and apply similarity in right triangles to understand right triangle trigonometry, with particular attention to special right triangles and the Pythagorean Theorem. Students develop facility with geometric proof. They use what they know about congruence and similarity to prove theorems involving lines, angles, triangles, and other polygons. They explore a variety of formats for writing proofs.

The Standards for Mathematical Practice complement the content standards so that students increasingly engage with the subject matter as they grow in mathematical maturity and expertise throughout the elementary, middle, and high school years.

Model Integrated Pathway: Model Mathematics II Overview [MII]

Number and Quantity

The Real Number System

- **A.** Extend the properties of exponents to rational exponents.
- **B.** Use properties of rational and irrational numbers.

Quantities

- **A.** Reason quantitatively and use units to solve problems.

The Complex Number Systems

- **A.** Perform arithmetic operations with complex numbers.
- **C.** Use complex numbers in polynomial identities and equations.

Algebra

Seeing Structure in Expressions

- **A.** Interpret the structure of quadratic and exponential expressions.
- **B.** Write quadratic and exponential expressions in equivalent forms to solve problems.

Arithmetic with Polynomials and Rational Expressions

- **A.** Perform arithmetic operations on polynomials.

Creating Equations

- **A.** Create equations that describe numbers or relationships.

Reasoning with Equations and Inequalities

- **B.** Solve equations and inequalities in one variable.
- **C.** Solve systems of equations.

Standards for Mathematical Practice

1. Make sense of problems and persevere in solving them.
2. Reason abstractly and quantitatively.
3. Construct viable arguments and critique the reasoning of others.
4. Model with mathematics.
5. Use appropriate tools strategically.
6. Attend to precision.
7. Look for and make use of structure.
8. Look for and express regularity in repeated reasoning.

Functions

Interpreting Functions

- **B.** Interpret quadratic and exponential functions with integer exponents that arise in applications in terms of the context.
- **C.** Analyze functions using different representations.

Building Functions

- **A.** Build a function that models a relationship between two quantities.
- **B.** Build new functions from existing functions.

Linear, Quadratic, and Exponential Models

- **A.** Construct and compare linear, quadratic and exponential models and solve problems.

Geometry

Congruence

- **C.** Prove geometric theorems, and when appropriate, the converse of theorems.

Similarity, Right Triangles, and Trigonometry

- **A.** Understand similarity in terms of similarity transformations.
- **B.** Prove theorems involving similarity using a variety of ways of writing proofs, showing validity of underlying reasoning.
- **C.** Define trigonometric ratios and solve problems involving right triangles.

Circles

- **A.** Understand and apply theorems about circles.
- **B.** Find arc lengths and areas of sectors of circles.

Expressing Geometric Properties with Equations

- **A.** Translate between the geometric description and the equation for a conic section.
- **B.** Use coordinates to prove simple geometric theorems algebraically.

Geometric Measurement and Dimension

- **A.** Explain volume formulas and use them to solve problems.

Statistics and Probability

Conditional Probability and the Rules of Probability

- **A.** Understand independence and conditional probability and use them to interpret data from simulations or experiments.
- **B.** Use the rules of probability to compute probabilities of compound events in a uniform probability model.

Model Integrated Pathway: Model Mathematics II Content Standards [MII]

Number and Quantity

The Real Number System MII.N-RN

A. Extend the properties of exponents to rational exponents.

1. Explain how the definition of the meaning of rational exponents follows from extending the properties of integer exponents to those values, allowing for a notation for radicals in terms of rational exponents.
   For example, we define $5^{1/3}$ to be the cube root of 5 because we want $(5^{1/3})3 = 5^{(1/3)3}$ to hold, so $(5^{1/3})^3$ must equal 5.
2. Rewrite expressions involving radicals and rational exponents using the properties of exponents.

B. Use properties of rational and irrational numbers.

3. Explain why the sum or product of two rational numbers is rational; that the sum of a rational number and an irrational number is irrational; and that the product of a nonzero rational number and an irrational number is irrational.

Quantities MII.N-Q

A. Reason quantitatively and use units to solve problems.

3. Choose a level of accuracy appropriate to limitations on measurement when reporting quantities.★
   a. Describe the effects of approximate error in measurement and rounding on measurements and on computed values from measurements. Identify significant figures in recorded measures and computed values based on the context given and the precision of the tools used to measure

The Complex Number System MII.N-CN

A. Perform arithmetic operations with complex numbers.

1. Know there is a complex number i such that $i^2 = -1$, and every complex number has the form $a + bi$ with a and b real.
2. Use the relation $i^2 = -1$ and the Commutative, Associative, and Distributive properties to add, subtract, and multiply complex numbers.

C. Use complex numbers in polynomial identities and equations.

7. Solve quadratic equations with real coefficients that have complex solutions.

Algebra

Seeing Structure in Expressions MII.A-SSE

A. Interpret the structure of quadratic and exponential expressions.

1. Interpret expressions that represent a quantity in terms of its context.★
   a. Interpret parts of an expression, such as terms, factors, and coefficients.
   b. Interpret complicated expressions by viewing one or more of their parts as a single entity.
   For example, interpret $P(1 + r)^t$ as the product of P and a factor not depending on P.
2. Use the structure of an expression to identify ways to rewrite it.
   For example, see $(x + 2)^2 – 9$ as a difference of squares that can be factored as $((x + 2) + 3)((x + 2) – 3)$.

B. Write quadratic and exponential expressions in equivalent forms to solve problems.

3. Choose and produce an equivalent form of an expression to reveal and explain properties of the quantity represented by the expression.
   a. Factor a quadratic expression to reveal the zeros of the function it defines.

- **b.** Complete the square in a quadratic expression to reveal the maximum or minimum value of the function it defines.
- **c.** Use the properties of exponents to transform expressions for exponential functions.

For example, the expression 1.15^t can be rewritten as $(1.15^{1/12})^{12t} \approx 1.012^{12t}$ to reveal the approximate equivalent monthly interest rate if the annual rate is 15%.

Arithmetic with Polynomials and Rational Expressions MII.A-APR

A. Perform arithmetic operations on polynomials.

1. Understand that polynomials form a system analogous to the integers, namely, they are closed under certain operations.
    - **a.** Perform operations on polynomial expressions (addition, subtraction, multiplication), and compare the system of polynomials to the system of integers when performing operations.
    - **b.** Factor and/or expand polynomial expressions; identify and combine like terms; and apply the Distributive property.

Creating Equations MII.A-CED

A. Create equations that describe numbers or relationships.

1. Create equations and inequalities in one variable and use them to solve problems. Include equations arising from quadratic and exponential functions.
2. Create equations in two or more variables to represent relationships between quantities; graph equations on coordinate axes with labels and scales.★
4. Rearrange formulas, including formulas with quadratic terms, to highlight a quantity of interest using the same reasoning as in solving equations (Properties of equality).

For example, rearrange Ohm's law $R = \frac{V^2}{P}$ to solve for voltage, V.

Reasoning with Equations and Inequalities MII.A-REI

B. Solve equations and inequalities in one variable.

4. Solve quadratic equations in one variable.
    - **a.** Use the method of completing the square to transform any quadratic equation in x into an equation of the form $(x - p)^2 = q$ that has the same solutions. Derive the quadratic formula from this form.
    - **b.** Solve quadratic equations by inspection (e.g., for $x^2 = 49$), taking square roots, completing the square, the quadratic formula, and factoring, as appropriate to the initial form of the equation. Recognize when the quadratic formula gives complex solutions and write them as $a \pm bi$ for real numbers a and b.

C. Solve systems of equations.

7. Solve a simple system consisting of a linear equation and a quadratic equation in two variables algebraically and graphically.

For example, find the points of intersection between the line $y = -3x$ and the circle $x^2 + y^2 = 3$.

Functions

Interpreting Functions MII.F-IF

B. Interpret quadratic and exponential functions with integer exponents that arise in applications in terms of the context.

4. For a function that models a relationship between two quantities, interpret key features of graphs and tables in terms of the quantities, and sketch graphs showing key features given a verbal description of

the relationship. Key features include: intercepts; intervals where the function is increasing, decreasing, positive, or negative; relative maximums and minimums; symmetries; and end behavior.★

5. Relate the domain of a function to its graph and, where applicable, to the quantitative relationship it describes.★

For example, if the function $h(n)$ gives the number of person-hours it takes to assemble n engines in a factory, then the positive integers would be an appropriate domain for the function.

6. Calculate and interpret the average rate of change of a function (presented symbolically or as a table) over a specified interval. Estimate the rate of change from a graph.★

C. Analyze functions using different representations.

7. Graph functions expressed symbolically and show key features of the graph, by hand in simple cases and using technology for more complicated cases.★

- **a.** Graph quadratic functions and show intercepts, maxima, and minima.★
- **b.** Graph piecewise-defined functions, including step functions and absolute value functions.★

8. Write a function defined by an expression in different but equivalent forms to reveal and explain different properties of the function.

- **a.** Use the process of factoring and completing the square in a quadratic function to show zeros, minimum/maximum values, and symmetry of the graph and interpret these in terms of a context.
- **b.** Use the properties of exponents to interpret expressions for exponential functions. Apply to financial situations such as Identifying appreciation/depreciation rate for the value of a house or car some time after its initial purchase: $V_n = P(1+r)^n$.

For example, identify percent rate of change in functions such as $y = (1.02)^t$, $y = (0.97)^t$, $y = (1.01)^{12t}$, and $y = (1.2)^{t/10}$, and classify them as representing exponential growth or decay.

9. Translate among different representations of functions (algebraically, graphically, numerically in tables, or by verbal descriptions). Compare properties of two functions each represented in a different way.

For example, given a graph of one quadratic function and an algebraic expression for another, say which has the larger maximum.

Building Functions MII.F-BF

A. Build a function that models a relationship between two quantities.

1. Write linear, quadratic, and exponential functions that describe relationships between two quantities.★

- **a.** Determine an explicit expression, a recursive process, or steps for calculation from a context.★
- **b.** Combine standard function types using arithmetic operations.★

For example, build a function that models the temperature of a cooling body by adding a constant function to a decaying exponential, and relate these functions to the model.

B. Build new functions from existing functions.

3. Identify the effect on the graph of replacing $f(x)$ by $f(x) + k$, $k\,f(x)$, $f(kx)$, and $f(x + k)$ for specific values of k (both positive and negative); find the value of k given the graphs. Include exponential, quadratic, and absolute value functions. Utilize technology to experiment with cases and illustrate an explanation of the effects on the graph.

4. Find inverse functions algebraically and graphically.

- **a.** Solve an equation of the form $f(x) = c$ for a linear function f that has an inverse and write an expression for the inverse.

Linear, Quadratic, and Exponential Models MII.F-LE

A. Construct and compare linear, quadratic, and exponential models and solve problems.

3. Observe using graphs and tables that a quantity increasing exponentially eventually exceeds a quantity increasing linearly, quadratically, or (more generally) as a polynomial function.★

Geometry

Congruence MII.G-CO

C. Prove geometric theorems and, when appropriate, the converse of theorems.

9. Prove theorems about lines and angles. Theorems include: vertical angles are congruent; when a transversal crosses parallel lines, alternate interior angles are congruent and corresponding angles are congruent, and conversely prove lines are parallel; points on a perpendicular bisector of a line segment are exactly those equidistant from the segment's endpoints.

10. Prove theorems about triangles. Theorems include: measures of interior angles of a triangle sum to 180°; base angles of isosceles triangles are congruent, and conversely prove a triangle is isosceles; the segment joining midpoints of two sides of a triangle is parallel to the third side and half the length; the medians of a triangle meet at a point.

11. Prove theorems about parallelograms. Theorems include: opposite sides are congruent, opposite angles are congruent, the diagonals of a parallelogram bisect each other, and conversely, rectangles are parallelograms with congruent diagonals.

- **a.** Prove theorems about polygons. Theorems include the measures of interior and exterior angles. Apply properties of polygons to the solutions of mathematical and contextual problems.

Similarity, Right Triangles, and Trigonometry MII.G-SRT

A. Understand similarity in terms of similarity transformations.

1. Verify experimentally the properties of dilations given by a center and a scale factor:

- **a.** A dilation takes a line not passing through the center of the dilation to a parallel line, and leaves a line passing through the center unchanged.
- **b.** The dilation of a line segment is longer or shorter in the ratio given by the scale factor.

2. Given two figures, use the definition of similarity in terms of similarity transformations to decide if they are similar; explain using similarity transformations the meaning of similarity for triangles as the equality of all corresponding pairs of angles and the proportionality of all corresponding pairs of sides.

3. Use the properties of similarity transformations to establish the Angle-Angle (AA) criterion for two triangles to be similar.

B. Prove theorems involving similarity using a variety of ways of writing proofs, showing validity of underlying reasoning.

4. Prove theorems about triangles. Theorems include: a line parallel to one side of a triangle divides the other two proportionally, and conversely; the Pythagorean Theorem proved using triangle similarity.

5. Use congruence and similarity criteria for triangles to solve problems and to prove relationships in geometric figures.

C. Define trigonometric ratios and solve problems involving right triangles.

6. Understand that by similarity, side ratios in right triangles are properties of the angles in the triangle, leading to definitions of trigonometric ratios for acute angles.

7. Explain and use the relationship between the sine and cosine of complementary angles.

8. Use trigonometric ratios and the Pythagorean Theorem to solve right triangles in applied problems.★

Circles MII.G-C

A. Understand and apply theorems about circles.

1. Prove that all circles are similar.

2. Identify and describe relationships among inscribed angles, radii, and chords. Include the relationship between central, inscribed, and circumscribed angles; inscribed angles on a diameter are right angles; the radius of a circle is perpendicular to the tangent where the radius intersects the circle.

3. Construct the inscribed and circumscribed circles of a triangle, and prove properties of angles for a quadrilateral and other polygons inscribed in a circle.

4. (+) Construct a tangent line from a point outside a given circle to the circle.

B. Find arc lengths and areas of sectors of circles.

5. Derive, using similarity, the fact that the length of the arc intercepted by an angle is proportional to the radius, and define the radian measure of the angle as the constant of proportionality; derive the formula for the area of a sector.

Expressing Geometric Properties with Equations MII.G-GPE

A. Translate between the geometric description and the equation for a conic section.

1. Derive the equation of a circle of given center and radius using the Pythagorean Theorem; complete the square to find the center and radius of a circle given by an equation.
2. Derive the equation of a parabola given a focus and directrix.

B. Use coordinates to prove simple geometric theorems algebraically.

4. Use coordinates to prove simple geometric theorems algebraically including the distance formula and its relationship to the Pythagorean Theorem.

   For example, prove or disprove that a figure defined by four given points in the coordinate plane is a rectangle; prove or disprove that the point (1, $\sqrt{3}$) lies on the circle centered at the origin and containing the point (0, 2).

6. Find the point on a directed line segment between two given points that partitions the segment in a given ratio.

Geometric Measurement and Dimension MII.G-GMD

A. Explain volume formulas and use them to solve problems.

1. Give an informal argument for the formulas for the circumference of a circle, area of a circle, volume of a cylinder, pyramid, and cone. Use dissection arguments, Cavalieri's principle, and informal limit arguments.
2. (+) Give an informal argument using Cavalieri's principle for the formulas for the volume of a sphere and other solid figures.
3. Use volume formulas for cylinders, pyramids, cones, and spheres to solve problems.★

Statistics and Probability

Conditional Probability and the Rules of Probability MII.S-CP

A. Understand independence and conditional probability and use them to interpret data from simulations or experiments.

1. Describe events as subsets of a sample space (the set of outcomes) using characteristics (or categories) of the outcomes, or as unions, intersections, or complements of other events ("or," "and," "not").★
2. Understand that two events *A* and *B* are independent if the probability of *A* and *B* occurring together is the product of their probabilities, and use this characterization to determine if they are independent.★
3. Understand the conditional probability of *A* given *B* as *P*(*A* and *B*)/*P*(*B*), and interpret independence of *A* and *B* as saying that the conditional probability of *A* given *B* is the same as the probability of *A*, and the conditional probability of *B* given *A* is the same as the probability of *B*.★
4. Construct and interpret two-way frequency tables of data when two categories are associated with each object being classified. Use the two-way table as a sample space to decide if events are independent and to approximate conditional probabilities.★

   For example, collect data from a random sample of students in your school on their favorite subject among math, science, and English. Estimate the probability that a randomly selected student from your school will favor science given that the student is in tenth grade. Do the same for other subjects and compare the results.

5. Recognize and explain the concepts of conditional probability and independence in everyday language and everyday situations.★

For example, compare the chance of having lung cancer if you are a smoker with the chance of being a smoker if you have lung cancer.

B. Use the rules of probability to compute probabilities of compound events in a uniform probability model.

6. Find the conditional probability of A given B as the fraction of B's outcomes that also belong to A, and interpret the answer in terms of the model.★

7. Apply the Addition Rule, $P(A \text{ or } B) = P(A) + P(B) - P(A \text{ and } B)$, and interpret the answer in terms of the model.★

8. (+) Apply the general Multiplication Rule in a uniform probability model, $P(A \text{ and } B) = P(A)P(B|A) = P(B)P(A|B)$, and interpret the answer in terms of the model.★

9. (+) Use permutations and combinations to compute probabilities of compound events and solve problems.★

Model Integrated Pathway: Model Mathematics III [MIII]

Introduction

It is in the Model Mathematics III course that students integrate and apply the mathematics they have learned from their earlier courses. For the high school Model Mathematics III course, instructional time should focus on four critical areas: (1) apply methods from probability and statistics to draw inferences and conclusions from data; (2) expand understanding of functions to include polynomial, rational, and radical functions;[26] (3) expand right triangle trigonometry to include general triangles; and (4) consolidate functions and geometry to create models and solve contextual problems.

1. Students see how the visual displays and summary statistics they learned in earlier grades relate to different types of data and to probability distributions. They identify different ways of collecting data—including sample surveys, experiments, and simulations—and the roles that randomness and careful design play in the conclusions that can be drawn.
2. The structural similarities between the system of polynomials and the system of integers are developed. Students draw on analogies between polynomial arithmetic and base-ten computation, focusing on properties of operations, particularly the distributive property. Students connect multiplication of polynomials with multiplication of multi-digit integers, and division of polynomials with long division of integers. Students identify zeros of polynomials and make connections between zeros of polynomials and solutions of polynomial equations. Rational numbers extend the arithmetic of integers by allowing division by all numbers except zero. Similarly, rational expressions extend the arithmetic of polynomials by allowing division by all polynomials except the zero polynomial. A central theme of the Model Mathematics III course is that the arithmetic of rational expressions is governed by the same rules as the arithmetic of rational numbers. This critical area also includes exploration of the Fundamental Theorem of Algebra.
3. Students derive the Laws of Sines and Cosines in order to find missing measures of general (not necessarily right) triangles. They are able to distinguish whether three given measures (angles or sides) define 0, 1, 2, or infinitely many triangles. This discussion of general triangles opens up the idea of trigonometry applied beyond the right triangle, at least to obtuse angles. Students build on this idea to develop the notion of radian measure for angles and extend the domain of the trigonometric functions to all real numbers. They apply this knowledge to model simple periodic phenomena.
4. Students synthesize and generalize what they have learned about a variety of function families. They extend their work with exponential functions to include solving exponential equations with logarithms. They explore the effects of transformations on graphs of diverse functions, including functions arising in an application, in order to abstract the general principle that transformations on a graph always have the same effect regardless of the type of underlying function. They identify appropriate types of functions to model a situation; they adjust parameters to improve the model; and they compare models by analyzing appropriateness of fit and making judgments about the domain over which a model is a good fit. The description of modeling as “the process of choosing and using mathematics and statistics to analyze empirical situations, to understand them better, and to make decisions” is at the heart of this Model Mathematics III course. The narrative discussion and diagram of the modeling cycle should be considered when knowledge of functions, statistics, and geometry is applied in a modeling context.

The Standards for Mathematical Practice complement the content standards so that students increasingly engage with the subject matter as they grow in mathematical maturity and expertise throughout the elementary, middle, and high school years.

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[26] In this course, rational functions are limited to those whose numerators are of degree at most 1 and denominators are of degree at most 2; radical functions are limited to square roots or cube roots of at most quadratic polynomials.

Model Integrated Pathway: Model Mathematics III Overview [MIII]

Number and Quantity

The Complex Number Systems

- **C.** Use complex numbers in polynomial identities and equations.

Vector and Matrix Quantities

- **A.** Represent and model with vector quantities.
- **C.** Perform operations on matrices and use matrices in applications.

Algebra

Seeing Structure in Expressions

- **A.** Interpret the structure polynomial and rational expressions.
- **B.** Write expressions in equivalent forms to solve problems.

Arithmetic with Polynomials and Rational Expressions

- **A.** Perform arithmetic operations on polynomials.
- **B.** Understand the relationship between zeros and factors of polynomials.
- **C.** Use polynomial identities to solve problems.
- **D.** Rewrite rational expressions.

Creating Equations

- **A.** Create equations that describe numbers or relationships.

Reasoning with Equations and Inequalities

- **A.** Understand solving equations as a process of reasoning and explain the reasoning.
- **D.** Represent and solve equations and inequalities graphically.

Functions

Interpreting Functions

- **B.** Interpret functions that arise in applications in terms of the context (rational, polynomial, square root, cube root, trigonometric, logarithmic).
- **C.** Analyze functions using different representations.

Building Functions

- **A.** Build a function that models a relationship between two quantities.
- **B.** Build new functions from existing functions.

Linear, Quadratic, and Exponential Models

- **A.** Construct and compare linear, quadratic and exponential models and solve problems.

Trigonometric Functions

- **A.** Extend the domain of trigonometric functions using the unit circle.
- **B.** Model periodic phenomena with trigonometric functions.
- **C.** Prove and apply trigonometric identities.

Standards for Mathematical Practice

1. Make sense of problems and persevere in solving them.
2. Reason abstractly and quantitatively.
3. Construct viable arguments and critique the reasoning of others.
4. Model with mathematics.
5. Use appropriate tools strategically.
6. Attend to precision.
7. Look for and make use of structure.
8. Look for and express regularity in repeated reasoning.

Geometry

Similarity, Right Triangles, and Trigonometry

- **D.** Apply trigonometry to general triangles.

Geometric Measurement and Dimension

- **B.** Visualize relationships between two-dimensional and three-dimensional objects.

Modeling with Geometry

- **A.** Apply geometric concepts in modeling situations.

Statistics and Probability

Interpreting Categorical and Quantitative Data

- **A.** Summarize, represent, and interpret data on a single count or measurement variable. Use calculators, spreadsheets, and other technology as appropriate.

Making Inferences and Justifying Conclusions

- **A.** Understand and evaluate random processes underlying statistical experiments.
- **B.** Make inferences and justify conclusions from sample surveys, experiments, and observational studies.

Using Probability to Make Decisions

- **B.** Use probability to evaluate outcomes of decisions.

Model Integrated Pathway: Model Mathematics III Content Standards [MIII]

Number and Quantity

The Complex Number System MIII.N-CN

C. Use complex numbers in polynomial identities and equations.

8. (+) Extend polynomial identities to the complex numbers.

For example, rewrite $x^2 + 4$ as $(x + 2i)(x – 2i)$.

9. (+) Know the Fundamental Theorem of Algebra; show that it is true for quadratic polynomials.

Vector and Matrix Quantities MIII.N-VM

A. Represent and model with vector quantities.

1. (+) Recognize vector quantities as having both magnitude and direction. Represent vector quantities by directed line segments, and use appropriate symbols for vectors and their magnitudes (e.g., $\mathbf{v}$, $|\mathbf{v}|$, $||\mathbf{v}||$, v).

3. (+) Solve problems involving velocity and other quantities that can be represented by vectors.

C. Perform operations on matrices and use matrices in applications.

6. (+) Use matrices to represent and manipulate data, e.g., to represent payoffs or incidence relationships in a network.

8. (+) Add, subtract, and multiply matrices of appropriate dimensions.

12. (+) Work with 2 × 2 matrices as transformations of the plane, and interpret the absolute value of the determinant in terms of area.

Algebra

Seeing Structure in Expressions MIII.A-SSE

A. Interpret the structure of polynomial and rational expressions.

1. Interpret expressions that represent a quantity in terms of its context.★

  a. Interpret parts of an expression, such as terms, factors, and coefficients.

  b. Interpret complicated expressions by viewing one or more of their parts as a single entity.

For example, interpret $P(1 + r)^n$ as the product of P and a factor not depending on P.

2. Use the structure of an expression to identify ways to rewrite it.

For example, see $x^4 – y^4$ as $(x^2)^2 – (y^2)^2$, thus recognizing it as a difference of squares that can be factored as $(x^2 – y^2)(x^2 + y^2)$ and as $(x-y)(x+y)(x-yi)(x+yi)$.

B. Write expressions in equivalent forms to solve problems.

4. Derive the formula for the sum of a finite geometric series (when the common ratio is not 1), and use the formula to solve problems.★

For example, calculate mortgage payments.

Arithmetic with Polynomials and Rational Expressions MIII.A-APR

A. Perform arithmetic operations on polynomials.

1. Understand that polynomials form a system analogous to the integers, namely, they are closed under certain operations.

  a. Perform operations on polynomial expressions (addition, subtraction, multiplication, and division), and compare the system of polynomials to the system of integers when performing operations.

B. Understand the relationship between zeros and factors of polynomials.

2. Know and apply the Remainder Theorem: For a polynomial $p(x)$ and a number a, the remainder on division by $x – a$ is $p(a)$, so $p(a) = 0$ if and only if $(x – a)$ is a factor of $p(x)$.
3. Identify zeros of polynomials when suitable factorizations are available, and use the zeros to construct a rough graph of the function defined by the polynomial.

C. Use polynomial identities to solve problems.

4. Prove polynomial identities and use them to describe numerical relationships.
   For example, the polynomial identity $(x^2 + y^2)^2 = (x^2 – y^2)^2 + (2xy)^2$ can be used to generate Pythagorean triples.
5. (+) Know and apply the Binomial Theorem for the expansion of $(x + y)^n$ in powers of x and y for a positive integer n, where x and y are any numbers, with coefficients determined for example by Pascal's Triangle.

D. Rewrite rational expressions.

6. Rewrite simple rational expressions in different forms; write $^{a(x)}/_{b(x)}$ in the form $q(x) + {}^{r(x)}/_{b(x)}$, where $a(x)$, $b(x)$, $q(x)$, and $r(x)$ are polynomials with the degree of $r(x)$ less than the degree of $b(x)$, using inspection, long division, or, for the more complicated examples, a computer algebra system.
7. (+) Understand that rational expressions form a system analogous to the rational numbers, closed under addition, subtraction, multiplication, and division by a nonzero rational expression; add, subtract, multiply, and divide rational expressions.

Creating Equations MIII.A-CED

A. Create equations that describe numbers or relationships.

1. Create equations and inequalities in one variable and use them to solve problems. (Include equations arising from simple root and rational functions.)★
2. Create equations in two or more variables to represent relationships between quantities; graph equations on coordinate axes with labels and scales.★
3. Represent constraints by equations or inequalities and by systems of equations and/or inequalities, and interpret solutions as viable or non-viable options in a modeling context.★
   For example, represent equations describing satellites orbiting earth and constraints on earth's size and atmosphere.

Reasoning with Equations and Inequalities MIII.A-REI

A. Understand solving equations as a process of reasoning and explain the reasoning.

2. Solve simple rational and radical equations in one variable, and give examples showing how extraneous solutions may arise.

D. Represent and solve equations and inequalities graphically.

11. Explain why the x-coordinates of the points where the graphs of the equations $y = f(x)$ and $y = g(x)$ intersect are the solutions of the equation $f(x) = g(x)$; find the solutions approximately, e.g., using technology to graph the functions, make tables of values, or find successive approximations. Include cases where $f(x)$ and/or $g(x)$ are polynomial, rational, and logarithmic functions.★

Functions

Interpreting Functions MIII.F-IF

B. Interpret functions that arise in applications in terms of the context (rational, polynomial, square root, cube root, trigonometric, logarithmic).

4. For a function that models a relationship between two quantities, interpret key features of graphs and tables in terms of the quantities, and sketch graphs showing key features given a verbal description of the relationship. Key features include: intercepts; intervals where the function is increasing, decreasing, positive, or negative; relative maximums and minimums; symmetries; end behavior; and periodicity.★
5. Relate the domain of a function to its graph and, where applicable, to the quantitative relationship it describes.★

For example, if the function h(n) gives the number of person-hours it takes to assemble n engines in a factory, then the positive integers would be an appropriate domain for the function.

6. Calculate and interpret the average rate of change of a function (presented symbolically or as a table) over a specified interval. Estimate the rate of change from a graph.★

C. Analyze functions using different representations.

7. Graph functions expressed symbolically and show key features of the graph, by hand in simple cases and using technology for more complicated cases.★
  - **b.** Graph square root and cube root functions.★
  - **c.** Graph polynomial functions, identifying zeros when suitable factorizations are available, and showing end behavior.★
  - **e.** Graph exponential and logarithmic functions, showing intercepts and end behavior, and trigonometric functions, showing period, midline, and amplitude.★

8. Write a function defined by an expression in different but equivalent forms to reveal and explain different properties of the function.
  - **a.** Use the process of factoring in a polynomial function to show zeros, extreme values, and symmetry of the graph, and interpret these in terms of a context.

9. Translate among different representations of functions: (algebraically, graphically, numerically in tables, or by verbal descriptions). Compare properties of two functions each represented in a different way.

For example, given a graph of one polynomial function and an algebraic expression for another, say which has the larger relative maximum and/or smaller relative minimum.

10. Given algebraic, numeric and/or graphical representations of functions, recognize the function as polynomial, rational, logarithmic, exponential, or trigonometric.

Building Functions MIII.F-BF

A. Build a function that models a relationship between two quantities.

1. Write simple rational and radical functions, logarithmic, and trigonometric functions that describes a relationship between two quantities.★
  - **b.** Combine standard function types using arithmetic operations.★

For example, build a function that models the temperature of a cooling body by adding a constant function to a decaying exponential, and relate these functions to the model (include simple rational and radical functions, logarithmic, and trigonometric functions).

B. Build new functions from existing functions.

3. Identify the effect on the graph of replacing $f(x)$ by $f(x) + k$, $kf(x)$, $f(kx)$, and $f(x + k)$ for specific values of k (both positive and negative); find the value of k given the graphs. Include simple rational, radical, logarithmic, and trigonometric functions. Experiment with cases and illustrate an explanation of the effects on the graph using technology. Include recognizing even and odd functions from their graphs and algebraic expressions for them.

4. Find inverse functions algebraically and graphically.
  - **a.** Solve an equation of the form $f(x) = c$ for a simple function f that has an inverse and write an expression for the inverse.

For example, $f(x) = 2x^3$ ***or*** $f(x) = {}^{(x+1)}/_{(x-1)}$ ***for*** $x \neq 1$**.**

Linear, Quadratic, and Exponential Models MIII.F-LE

A. Construct and compare linear, quadratic, and exponential models and solve problems.

4. For exponential models, express as a logarithm the solution to $ab^{ct} = d$ where a, c, and d are numbers and the base b is 2, 10, or e; evaluate the logarithm using technology.★

Trigonometric Functions MIII.F-TF

A. Extend the domain of trigonometric functions using the unit circle.

1. Understand radian measure of an angle as the length of the arc on the unit circle subtended by the angle.
2. Explain how the unit circle in the coordinate plane enables the extension of trigonometric functions to all real numbers, interpreted as radian measures of angles traversed counterclockwise around the unit circle.

B. Model periodic phenomena with trigonometric functions.

5. Choose trigonometric functions to model periodic phenomena with specified amplitude, frequency, and midline.★

C. Prove and apply trigonometric identities.

8. Prove the Pythagorean identity $\sin^2(\theta) + \cos^2(\theta) = 1$ and use it to find sin(θ), cos(θ), or tan(θ) given sin(θ), cos(θ), or tan(θ) and the quadrant.

Geometry

Similarity, Right Triangles, and Trigonometry MIII.G-SRT

D. Apply trigonometry to general triangles.

9. (+) Derive the formula $A = ½ ab \sin(C)$ for the area of a triangle by drawing an auxiliary line from a vertex perpendicular to the opposite side.
10. (+) Prove the Laws of Sines and Cosines and use them to solve problems.
11. (+) Understand and apply the Law of Sines and the Law of Cosines to find unknown measurements in right and non-right triangles (e.g., surveying problems, resultant forces).

Geometric Measurement and Dimension MIII.G-GMD

B. Visualize relationships between two-dimensional and three-dimensional objects.

4. Identify the shapes of two-dimensional cross-sections of three-dimensional objects, and identify three-dimensional objects generated by rotations of two-dimensional objects.

Modeling with Geometry MIII.G-MG

A. Apply geometric concepts in modeling situations.

1. Use geometric shapes, their measures, and their properties to describe objects (e.g., modeling a tree trunk or a human torso as a cylinder).★
2. Apply concepts of density based on area and volume in modeling situations (e.g., persons per square mile, BTUs per cubic foot).★
3. Apply geometric methods to solve design problems (e.g., designing an object or structure to satisfy physical constraints or minimize cost; working with typographic grid systems based on ratios).★
4. Use dimensional analysis for unit conversions to confirm that expressions and equations make sense.★

Statistics and Probability

Interpreting Categorical and Quantitative Data MIII.S-ID

A. Summarize, represent, and interpret data on a single count or measurement variable. Use calculators, spreadsheets, and other technology as appropriate.

4. Use the mean and standard deviation of a data set to fit it to a normal distribution and to estimate population percentages. Recognize that there are data sets for which such a procedure is not appropriate. Use calculators, spreadsheets, and tables to estimate areas under the normal curve.★

Making Inferences and Justifying Conclusions MIII.S-IC

A. Understand and evaluate random processes underlying statistical experiments.

1. Understand statistics as a process for making inferences to be made about population parameters based on a random sample from that population.★
2. Decide if a specified model is consistent with results from a given data-generating process, e.g., using simulation.★

   For example, a model says a spinning coin falls heads up with probability 0.5. Would a result of five tails in a row cause you to question the model?

B. Make inferences and justify conclusions from sample surveys, experiments, and observational studies.

3. Recognize the purposes of and differences among sample surveys, experiments, and observational studies; explain how randomization relates to each.★
4. Use data from a sample survey to estimate a population mean or proportion; develop a margin of error through the use of simulation models for random sampling.★
5. Use data from a randomized experiment to compare two treatments; use simulations to decide if differences between parameters are significant.★
6. Evaluate reports based on data.★

Using Probability to Make Decisions MIII.S-MD

B. Use probability to evaluate outcomes of decisions.

6. (+) Use probabilities to make fair decisions (e.g., drawing by lots, using a random number generator).★
7. (+) Analyze decisions and strategies using probability concepts (e.g., product testing, medical testing, pulling a hockey goalie at the end of a game and replacing the goalie with an extra skater).★

Model Advanced Course: Model Precalculus [PC]

Introduction

Precalculus combines the trigonometric, geometric, and algebraic techniques needed to prepare students for the study of calculus, and strengthens students' conceptual understanding of problems and mathematical reasoning in solving problems. Facility with these topics is especially important for students intending to study calculus, physics, and other sciences, and/or engineering in college. Because the standards for this course are (+) standards, students selecting this Model Precalculus course should have met the college and career ready standards.

For the high school Model Precalculus course, instructional time should focus on four critical areas: (1) extend work with complex numbers; (2) expand understanding of logarithms and exponential functions; (3) use characteristics of polynomial and rational functions to sketch graphs of those functions; and (4) perform operations with vectors.

1. Students continue their work with complex numbers. They perform arithmetic operations with complex numbers and represent them and the operations on the complex plane. Students investigate and identify the characteristics of the graphs of polar equations, using graphing tools. This includes classification of polar equations; the effects of changes in the parameters in polar equations; conversion of complex numbers from rectangular form to polar form and vice versa; and the intersection of the graphs of polar equations.
2. Students expand their understanding of functions to include logarithmic and trigonometric functions. They investigate and identify the characteristics of exponential and logarithmic functions in order to graph these functions and solve equations and practical problems. This includes the role of *e*, natural and common logarithms, laws of exponents and logarithms, and the solutions of logarithmic and exponential equations. Students model periodic phenomena with trigonometric functions and prove trigonometric identities. Other trigonometric topics include reviewing unit circle trigonometry, proving trigonometric identities, solving trigonometric equations, and graphing trigonometric functions.
3. Students investigate and identify the characteristics of polynomial and rational functions and use these to sketch the graphs of the functions. They determine zeros, upper and lower bounds, y-intercepts, symmetry, asymptotes, intervals for which the function is increasing or decreasing, and maximum or minimum points. Students translate between the geometric description and equation of conic sections. They deepen their understanding of the Fundamental Theorem of Algebra.
4. Students perform operations with vectors in the coordinate plane and solve practical problems using vectors. This includes the following topics: operations of addition, subtraction, scalar multiplication, and inner (dot) product; norm of a vector; unit vector; graphing; properties; simple proofs; complex numbers (as vectors); and perpendicular components.

The Standards for Mathematical Practice complement the content standards so that students increasingly engage with the subject matter as they grow in mathematical maturity and expertise throughout the elementary, middle, and high school years.

Model Advanced Course: Model Precalculus Overview [PC]

Number and Quantity

The Complex Number System

- **A.** Perform arithmetic operations with complex numbers.
- **B.** Represent complex numbers and their operations on the complex plane.
- **C.** Use complex numbers in polynomial identities and equations.

Vector and Matrix Quantities

- **A.** Represent and model with vector quantities.
- **B.** Perform operations on vectors.
- **C.** Perform operations on matrices and use matrices in applications.

Algebra

Arithmetic with Polynomials and Rational Expressions

- **C.** Use polynomial identities to solve problems
- **D.** Rewrite rational expressions.

Reasoning with Equations and Inequalities

- **C.** Solve systems of equations.

Functions

Interpreting Functions

- **C.** Analyze functions using different representations.

Building Functions

- **A.** Build a function that models a relationship between two quantities.
- **B.** Build new functions from existing functions.

Trigonometric Functions

- **A.** Extend the domain of trigonometric functions using the unit circle.
- **B.** Model periodic phenomena with trigonometric functions.
- **C.** Prove and apply trigonometric identities.

Geometry

Similarity, Right Triangles, and Trigonometry

- **D.** Apply trigonometry to general triangles.

Circles

- **A.** Understand and apply theorems about circles.

Expressing Geometric Properties with Equations

- **A.** Translate between the geometric description and the equation for a conic section.

Geometric Measurement and Dimension

- **A.** Explain volume formulas and use them to solve problems.

Standards for Mathematical Practice

1. Make sense of problems and persevere in solving them.
2. Reason abstractly and quantitatively.
3. Construct viable arguments and critique the reasoning of others.
4. Model with mathematics.
5. Use appropriate tools strategically.
6. Attend to precision.
7. Look for and make use of structure.
8. Look for and express regularity in repeated reasoning.

Model Advanced Course: Model Precalculus Content Standards [PC]

Number and Quantity

The Complex Number System PC.N-CN

A. Perform arithmetic operations with complex numbers.

3. (+) Find the conjugate of a complex number; use conjugates to find moduli and quotients of complex numbers.

B. Represent complex numbers and their operations on the complex plane.

4. (+) Represent complex numbers on the complex plane in rectangular and polar form (including real and imaginary numbers), and explain why the rectangular and polar forms of a given complex number represent the same number.

5. (+) Represent addition, subtraction, multiplication, and conjugation of complex numbers geometrically on the complex plane; use properties of this representation for computation.

For example, $(-1+\sqrt{3i})^3=8$ ***because*** $(-1+\sqrt{3i})$ ***has modulus 2 and argument 120°.***

6. (+) Calculate the distance between numbers in the complex plane as the modulus of the difference, and the midpoint of a segment as the average of the numbers at its endpoints.

C. Use complex numbers in polynomial identities and equations.

8. (+) Extend polynomial identities to the complex numbers.

For example, rewrite $x^2 + 4$ ***as*** $(x + 2i)(x - 2i)$.

9. (+) Know the Fundamental Theorem of Algebra; show that it is true for quadratic polynomials.

Vector and Matrix Quantities PC.N-VM

A. Represent and model with vector quantities.

1. (+) Recognize vector quantities as having both magnitude and direction. Represent vector quantities by directed line segments, and use appropriate symbols for vectors and their magnitudes (e.g., $\boldsymbol{v}$, $|\boldsymbol{v}|$, $||\boldsymbol{v}||$, v).

2. (+) Find the components of a vector by subtracting the coordinates of an initial point from the coordinates of a terminal point.

3. (+) Solve problems involving velocity and other quantities that can be represented by vectors.

B. Perform operations on vectors.

4. (+) Add and subtract vectors.

- **a.** (+) Add vectors end-to-end, component-wise, and by the parallelogram rule. Understand that the magnitude of a sum of two vectors is typically not the sum of the magnitudes.
- **b.** (+) Given two vectors in magnitude and direction form, determine the magnitude and direction of their sum.
- **c.** (+) Understand vector subtraction $\boldsymbol{v} - \boldsymbol{w}$ as $\boldsymbol{v} + (-\boldsymbol{w})$, where $-\boldsymbol{w}$ is the additive inverse of $\boldsymbol{w}$, with the same magnitude as $\boldsymbol{w}$ and pointing in the opposite direction. Represent vector subtraction graphically by connecting the tips in the appropriate order, and perform vector subtraction component-wise.

5. (+) Multiply a vector by a scalar.

- **a.** (+) Represent scalar multiplication graphically by scaling vectors and possibly reversing their direction; perform scalar multiplication component-wise, e.g., as $c(v_x, v_y) = (cv_x, cv_y)$.
- **b.** (+) Compute the magnitude of a scalar multiple $c\boldsymbol{v}$ using $||c\boldsymbol{v}|| = |c|\boldsymbol{v}$. Compute the direction of $c\boldsymbol{v}$ knowing that when $|c|\boldsymbol{v} \neq 0$, the direction of $c\boldsymbol{v}$ is either along $\boldsymbol{v}$ (for $c > 0$) or against $\boldsymbol{v}$ (for $c < 0$).

C. Perform operations on matrices and use matrices in applications.

6. (+) Use matrices to represent and manipulate data, e.g., to represent payoffs or incidence relationships in a network.
7. (+) Multiply matrices by scalars to produce new matrices, e.g., as when all of the payoffs in a game are doubled.
8. (+) Add, subtract, and multiply matrices of appropriate dimensions.
9. (+) Understand that, unlike multiplication of numbers, matrix multiplication for square matrices is not a Commutative operation, but still satisfies the Associative and Distributive properties.
10. (+) Understand that the zero and identity matrices play a role in matrix addition and multiplication similar to the role of 0 and 1 in the real numbers. The determinant of a square matrix is nonzero if and only if the matrix has a multiplicative inverse.
11. (+) Multiply a vector (regarded as a matrix with one column) by a matrix of suitable dimensions to produce another vector. Work with matrices as transformations of vectors.
12. (+) Work with 2 × 2 matrices as transformations of the plane, and interpret the absolute value of the determinant in terms of area.

Algebra

Arithmetic with Polynomials and Rational Expressions PC.A-APR

C. Use polynomial identities to solve problems.

5. (+) Know and apply the Binomial Theorem for the expansion of $(x + y)^n$ in powers of x and y for a positive integer n, where x and y are any numbers, with coefficients determined for example by Pascal's Triangle.[27]

D. Rewrite rational expressions.

7. (+) Understand that rational expressions form a system analogous to the rational numbers, closed under addition, subtraction, multiplication, and division by a nonzero rational expression; add, subtract, multiply, and divide rational expressions.

Reasoning with Equations and Inequalities PC.A-REI

C. Solve systems of equations.

8. (+) Represent a system of linear equations as a single matrix equation in a vector variable.
9. (+) Find the inverse of a matrix if it exists and use it to solve systems of linear equations (using technology for matrices of dimension 3 × 3 or greater).

Functions

Interpreting Functions PC.F-IF

C. Analyze functions using different representations.

7. Graph functions expressed symbolically and show key features of the graph, by hand in simple cases and using technology for more complicated cases.★
    - **d.** (+) Graph rational functions, identifying zeros and asymptotes when suitable factorizations are available, and showing end behavior.★

Building Functions PC.F-BF

A. Build a function that models a relationship between two quantities.

1. Write a function that describes a relationship between two quantities.★
    - **c.** (+) Compose functions. ★

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[27] The Binomial Theorem can be proved by mathematical induction or by a combinatorial argument.

For example, if $T(y)$ is the temperature in the atmosphere as a function of height, and $h(t)$ is the height of a weather balloon as a function of time, then $T(h(t))$ is the temperature at the location of the weather balloon as a function of time.

B. Build new functions from existing functions.

- **4.** Find inverse functions.
  - **b.** (+) Verify by composition that one function is the inverse of another.
  - **c.** (+) Read values of an inverse function from a graph or a table, given that the function has an inverse.
  - **d.** (+) Produce an invertible function from a non-invertible function by restricting the domain.
- **5.** (+) Understand the inverse relationship between exponents and logarithms and use this relationship to solve problems involving logarithms and exponents.

Trigonometric Functions PC.F-TF

A. Extend the domain of trigonometric functions using the unit circle.

- **3.** (+) Use special triangles to determine geometrically the values of sine, cosine, tangent for $^{\pi}/_{3}$, $^{\pi}/_{4}$ and $^{\pi}/_{6}$, and use the unit circle to express the values of sine, cosine, and tangent for $\pi - x$, $\pi + x$, and $2\pi - x$ in terms of their values for x, where x is any real number.
- **4.** (+) Use the unit circle to explain symmetry (odd and even) and periodicity of trigonometric functions.

B. Model periodic phenomena with trigonometric functions.

- **6.** (+) Understand that restricting a trigonometric function to a domain on which it is always increasing or always decreasing allows its inverse to be constructed.
- **7.** (+) Use inverse functions to solve trigonometric equations that arise in modeling contexts; evaluate the solutions using technology, and interpret them in terms of the context.★

C. Prove and apply trigonometric identities.

- **9.** (+) Prove the addition and subtraction formulas for sine, cosine, and tangent and use them to solve problems.

Geometry

Similarity, Right Triangles, and Trigonometry PC.G-SRT

D. Apply trigonometry to general triangles.

- **9.** (+) Derive the formula $A = \frac{1}{2} ab \sin(C)$ for the area of a triangle by drawing an auxiliary line from a vertex perpendicular to the opposite side.
- **10.** (+) Prove the Laws of Sines and Cosines and use them to solve problems.
- **11.** (+) Understand and apply the Law of Sines and the Law of Cosines to find unknown measurements in right and non-right triangles (e.g., surveying problems, resultant forces).

Circles PC.G-C

A. Understand and apply theorems about circles.

- **4.** (+) Construct a tangent line from a point outside a given circle to the circle.

Expressing Geometric Properties with Equations PC.G-GPE

A. Translate between the geometric description and the equation for a conic section.

- **3.** (+) Derive the equations of ellipses and hyperbolas given the foci, using the fact that the sum or difference of distances from the foci is constant.
  - **a.** (+) Use equations and graphs of conic sections to model real-world problems.★

Geometric Measurement and Dimension PC.G-GMD

A. Explain volume formulas and use them to solve problems.

2. (+) Give an informal argument using Cavalieri’s principle for the formulas for the volume of a sphere and other solid figures.

Model Advanced Course: Model Advanced Quantitative Reasoning [AQR]

Introduction

Because the standards for this course are (+) standards, students selecting this Model Advanced Quantitative Reasoning course should have met the college and career ready standards.

The high school Model Advanced Quantitative Reasoning course is designed as a mathematics course alternative to Precalculus. Through this course, students are encouraged to continue their study of mathematical ideas in the context of real-world problems and decision making through the analysis of information, modeling change, and mathematical relationships.

For the high school Model Advanced Quantitative Reasoning course, instructional time should focus on three critical areas: (1) critique quantitative data; (2) investigate and apply various mathematical models; and (3) explore and apply concepts of vectors and matrices to model and solve real-world problems.

1. Students learn to become critical consumers of the quantitative data that surround them every day, knowledgeable decision makers who use logical reasoning, and mathematical thinkers who can use their quantitative skills to solve problems related to a wide range of situations. They link classroom mathematics and statistics to everyday life, work, and decision making, using mathematical modeling. They choose and use appropriate mathematics and statistics to analyze empirical situations, to understand them better, and to improve decisions.
2. Through the investigation of mathematical models from real-world situations, students strengthen conceptual understandings in mathematics and further develop connections between algebra and geometry. Students use geometry to model real-world problems and solutions. They use the language and symbols of mathematics in representations and communication.
3. Students explore linear algebra concepts of matrices and vectors. They use vectors to model physical relationships to define and solve real-world problems. Students draw, name, label, and describe vectors, perform operations with vectors, and relate these components to vector magnitude and direction. They use matrices in relationship to vectors and to solve problems.

The Standards for Mathematical Practice complement the content standards so that students increasingly engage with the subject matter as they grow in mathematical maturity and expertise throughout the elementary, middle, and high school years.

Model Advanced Course: Model Advanced Quantitative Reasoning Overview [AQR]

Number and Quantity

Vector and Matrix Quantities

- **A.** Represent and model with vector quantities.
- **C.** Perform operations on matrices and use matrices in applications.

Algebra

Arithmetic with Polynomials and Rational Expressions

- **C.** Use polynomials identities to solve problems.

Reasoning with Equations and Inequalities

- **C.** Solve systems of equations.

Functions

Trigonometric Functions

- **A.** Extend the domain of trigonometric functions using the unit circle.
- **B.** Model periodic phenomena with trigonometric functions.
- **C.** Prove and apply trigonometric identities.

Geometry

Similarity, Right Triangles, and Trigonometry

- **D.** Apply trigonometry to general triangles.

Circles

- **A.** Understand and apply theorems about circles.

Expressing Geometric Properties with Equations

- **A.** Translate between the geometric description and the equation for a conic section.

Geometric Measurement and Dimension

- **A.** Explain volume formulas and use them to solve problems.

Statistics and Probability

Conditional Probability and the Rules of Probability

- **B.** Use the rules of probability to compute probabilities of compound events in a uniform probability model.

Using Probability to Make Decisions

- **A.** Calculate expected values and use them to solve problems.
- **B.** Use probability to evaluate outcomes of decisions.

Standards for Mathematical Practice

1. Make sense of problems and persevere in solving them.
2. Reason abstractly and quantitatively.
3. Construct viable arguments and critique the reasoning of others.
4. Model with mathematics.
5. Use appropriate tools strategically.
6. Attend to precision.
7. Look for and make use of structure.
8. Look for and express regularity in repeated reasoning.

Model Advanced Course: Model Advanced Quantitative Reasoning Content Standards [AQR]

Number and Quantity

Vector and Matrix Quantities AQR.N-VM

A. Represent and model with vector quantities.

1. (+) Recognize vector quantities as having both magnitude and direction. Represent vector quantities by directed line segments, and use appropriate symbols for vectors and their magnitudes (e.g., $\boldsymbol{v}$, $|\boldsymbol{v}|$, $||\boldsymbol{v}||$, v).
2. (+) Find the components of a vector by subtracting the coordinates of an initial point from the coordinates of a terminal point.
3. (+) Solve problems involving velocity and other quantities that can be represented by vectors.

C. Perform operations on matrices and use matrices in applications.

6. (+) Use matrices to represent and manipulate data, e.g., to represent payoffs or incidence relationships in a network.
7. (+) Multiply matrices by scalars to produce new matrices, e.g., as when all of the payoffs in a game are doubled.
8. (+) Add, subtract, and multiply matrices of appropriate dimensions.
9. (+) Understand that, unlike multiplication of numbers, matrix multiplication for square matrices is not a Commutative operation, but still satisfies the Associative and Distributive properties.
10. (+) Understand that the zero and identity matrices play a role in matrix addition and multiplication similar to the role of 0 and 1 in the real numbers. The determinant of a square matrix is nonzero if and only if the matrix has a multiplicative inverse.
11. (+) Multiply a vector (regarded as a matrix with one column) by a matrix of suitable dimensions to produce another vector. Work with matrices as transformations of vectors.
12. (+) Work with 2×2 matrices as transformations of the plane, and interpret the absolute value of the determinant in terms of area.

Algebra

Arithmetic with Polynomials and Rational Expressions AQR.A-APR

C. Use polynomial identities to solve problems.

5. (+) Know and apply the Binomial Theorem for the expansion of $(x + y)^n$ in powers of x and y for a positive integer n, where x and y are any numbers, with coefficients determined for example by Pascal's Triangle.[28]

Reasoning with Equations and Inequalities AQR.A-REI

C. Solve systems of equations.

8. (+) Represent a system of linear equations as a single matrix equation in a vector variable.
9. (+) Find the inverse of a matrix if it exists and use it to solve systems of linear equations (using technology for matrices of dimension 3×3 or greater).

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[28] The Binomial Theorem can be proved by mathematical induction or by a combinatorial argument.

Functions

Trigonometric Functions AQR.F-TF

A. Extend the domain of trigonometric functions using the unit circle.

3. (+) Use special triangles to determine geometrically the values of sine, cosine, tangent for $\pi/3$, $\pi/4$ and $\pi/6$, and use the unit circle to express the values of sine, cosine, and tangent for $\pi - x$, $\pi + x$, and $2\pi - x$ in terms of their values for x, where x is any real number.
4. (+) Use the unit circle to explain symmetry (odd and even) and periodicity of trigonometric functions.

B. Model periodic phenomena with trigonometric functions.

7. (+) Use inverse functions to solve trigonometric equations that arise in modeling contexts; evaluate the solutions using technology, and interpret them in terms of the context.★

C. Prove[29] and apply trigonometric identities.

9. (+) Prove the addition and subtraction formulas for sine, cosine, and tangent and use them to solve problems.

Geometry

Similarity, Right Triangles, and Trigonometry AQR.G-SRT

D. Apply trigonometry to general triangles.

11. (+) Understand and apply the Law of Sines and the Law of Cosines to find unknown measurements in right and non-right triangles (e.g., surveying problems, resultant forces).

Circles AQR.G-C

A. Understand and apply theorems about circles.

4. (+) Construct a tangent line from a point outside a given circle to the circle.

Expressing Geometric Properties with Equations AQR.G-GPE

A. Translate between the geometric description and the equation for a conic section.

3. (+) Derive the equations of ellipses and hyperbolas given the foci, using the fact that the sum or difference of distances from the foci is constant.
    - **a.** (+) Use equations and graphs of conic sections to model real-world problems.★

Geometric Measurement and Dimension AQR.G-GMD

A. Explain volume formulas and use them to solve problems.

2. (+) Give an informal argument using Cavalieri's principle for the formulas for the volume of a sphere and other solid figures.

Statistics and Probability

Conditional Probability and the Rules of Probability AQR.S-CP

B. Use the rules of probability to compute probabilities of compound events in a uniform probability model.

8. (+) Apply the general Multiplication Rule in a uniform probability model, $P(A \text{ and } B) = P(A)P(B|A) = P(B)P(A|B)$, and interpret the answer in terms of the model.★

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[29] Advanced Quantitative Reasoning should accept informal proof and focus on the underlying reasoning, and use the theorems to solve problems.

9. (+) Use permutations and combinations to compute probabilities of compound events and solve problems.★

Using Probability to Make Decisions AQR.S-MD

A. Calculate expected values and use them to solve problems.

1. (+) Define a random variable for a quantity of interest by assigning a numerical value to each event in a sample space; graph the corresponding probability distribution using the same graphical displays as for data distributions.★
2. (+) Calculate the expected value of a random variable; interpret it as the mean of the probability distribution.★
3. (+) Develop a probability distribution for a random variable defined for a sample space in which theoretical probabilities can be calculated; find the expected value.★

   For example, find the theoretical probability distribution for the number of correct answers obtained by guessing on all five questions of a multiple-choice test where each question has four choices, and find the expected grade under various grading schemes.
4. (+) Develop a probability distribution for a random variable defined for a sample space in which probabilities are assigned empirically; find the expected value.★

   For example, find a current data distribution on the number of TV sets per household in the United States, and calculate the expected number of sets per household. How many TV sets would you expect to find in 100 randomly selected households?

B. Use probability to evaluate outcomes of decisions.

5. (+) Weigh the possible outcomes of a decision by assigning probabilities to payoff values and finding expected values.★
   - **a.** (+) Find the expected payoff for a game of chance.★

     For example, find the expected winnings from a state lottery ticket or a game at a fast-food restaurant.
   - **b.** (+) Evaluate and compare strategies on the basis of expected values.★

     For example, compare a high-deductible versus a low-deductible automobile insurance policy using various, but reasonable, chances of having a minor or a major accident.
6. (+) Use probabilities to make fair decisions (e.g., drawing by lots, using a random number generator).★
7. (+) Analyze decisions and strategies using probability concepts (e.g., product testing, medical testing, pulling a hockey goalie at the end of a game and replacing the goalie with an extra skater).

Making Decisions about High School Course Sequences and Algebra I in Grade 8

Course Sequences and the Model Algebra I Course

The 2017 *Massachusetts Curriculum Framework for Mathematics* represents an opportunity to revisit course sequences in middle and high school mathematics. Districts should work with stakeholders, including middle and high school teachers, guidance counselors, external partners and families, to systematically consider the full range of considerations related to course offerings and sequencing in mathematics in light of these revised standards.

Students who follow the grade-by-grade pre-kindergarten to grade 8 sequence will be prepared for either the Traditional or Integrated Model Course high school pathways beginning with Algebra I or Mathematics I in grade 9 and will be ready to take a fourth year advanced course in grade 12, such as the Model Precalculus Course, the Model Quantitative Reasoning Course, or other advanced courses offered in the district, such as Statistics.

Decisions about secondary students' course-taking sequences should be made with the goal of identifying each student's path to success and ensuring that **no** student who graduates from a Massachusetts High School and enrolls in a Massachusetts public college or university will be placed into a non-credit bearing remedial mathematics course.

All students should be encouraged to meet the full expectations of the pre-K to high school standards. There should also be a variety of ways and opportunities for students to take advanced mathematics courses beyond those included in this Framework. Districts are encouraged to work with their mathematics teachers and curriculum coordinators to design pathways that best meet the needs of their students.

This section presents information and resources to ground discussions and decision-making about course-taking sequences in three inter-related areas of consideration:

- The rigor of the grades 6–8 standards and the Model High School Algebra I Course standards.
- The offering of the Model High School Algebra I Course in grade 8 for students for whom it is appropriate.
- Options for high school pathways that accelerate starting in grade 9 to allow students to reach advanced mathematics courses, such as Calculus in grade 12.

I. Rigor of Grade 8 and the Model High School Algebra I Standards

Success in Algebra I is crucial to students' overall academic success and their continued interest and engagement in mathematics. The pre-kindergarten to grade 8 standards in the 2017 Framework present a tight progression of skills and knowledge that is rigorous and designed to provide a strong foundation for success in Algebra I as defined in the High School Model Algebra I Course.

Course Sequences and the Model Algebra I Course

The grade 8 standards address foundations of algebra, a more formal treatment of functions, the exploration of irrational numbers and the Pythagorean Theorem; and include geometry standards that relate graphing to algebra and statistics concepts; and skills that are sophisticated and connect linear relations with the representation of bivariate data.

The Model Algebra I course formalizes and builds on the grade 8 standards. This course begins with more advanced topics and deepens and extends students' understanding of linear functions, exponential functions and relationships, introduces quadratic relationships, and includes rigorous statistics concepts and skills.

II. Offering the Model High School Algebra I course in middle school to grade 8 students for whom it is appropriate (Compacted Pathway)

The Mathematics Standards in grades 6–8 are coherent, rigorous, and non-redundant, so the offering of high school coursework in middle school to students for whom it is appropriate requires careful planning to ensure that all content and practice standards are fully addressed. For those students ready to move at a more accelerated pace, one option is to compress the standards for any three consecutive grades and/or courses into an accelerated two-year pathway.

Compressing the standards from grade 7, grade 8, and the Model Algebra I (or Model Mathematics I) course into an accelerated pathway for students in grades 7 and 8 could allow students to enter the Model Geometry (or Model Mathematics II) course in grade 9.

Selecting and placing students into accelerated opportunities must be done carefully in order to ensure success. Students who follow a compacted pathway will be undertaking advanced work at an accelerated pace. This creates a challenge for these students as well as their teachers, who will be teaching the grade 8 standards and Model Algebra I standards within a compressed timeframe without compromising any of the rigor. Placement decisions should be made based upon a common assessment to be reviewed by a team of stakeholders that includes teachers and administrators.

III. Accelerated High School Pathways starting in grade 9 to allow students to reach advanced mathematics courses, such as Calculus, by grade 12

For many students, high school mathematics will culminate during grade 12 with courses such as Model Precalculus and/or Model Advanced Quantitative Reasoning. Although this would represent a robust and rigorous course of study, some students will seek the opportunity to advance to mathematics courses beyond those included in this Framework (for example, Discrete Mathematics, Linear Algebra, AP Statistics and/or AP Calculus). The following models are only some of the pathways by which students' mathematical needs could be met. Districts are encouraged to work with their mathematics administrators, teachers, and curriculum coordinators to design pathways that best meet the abilities and needs of their students.

In high school, compressed and accelerated pathways may follow these models, among others:

- Students could "double up" by enrolling in the Model Geometry course during the same year as Model Algebra I or Model Algebra II.
- Standards from the Model Precalculus course could be added to other courses in a high school pathway, allowing students to enter a Calculus course without enrolling in the Model Precalculus course.
- Standards that focus on a sub-topic such as trigonometry or statistics could be pulled out and taken alongside the Model courses so that students would only need to "double up" for one semester.
- Standards from the Model Mathematics I, Model Mathematics II, and Mathematics III course could be compressed into an accelerated pathway for students for two years, allowing students to enter the Model Precalculus course in the third year.

The graphics below depict options for grades 6–12 course sequences:

Figure 1 shows a 6–8 grade-by-grade progression followed by the three Model High School Courses culminating in an advanced mathematics course in grade 12.

Figure 2–4 depicts three accelerated pathways leading to Calculus. The first accelerated pathway in **Figure 2** compresses grades 7, 8, and the High School Model Algebra I course standards in two years. This compacting of standards begins during middle school at the end of grade 6 and ends with Algebra I in grade 8.

The last two pathways in **Figure 3** and **Figure 4** are high school accelerated pathway options, titled "Doubling Up" and "Enhanced Pathway." Note that the accelerated high school pathways delay decisions about accelerating students until they are in high school while still allowing access to advanced mathematics in grade 12.

Figure 1: On grade level pathway

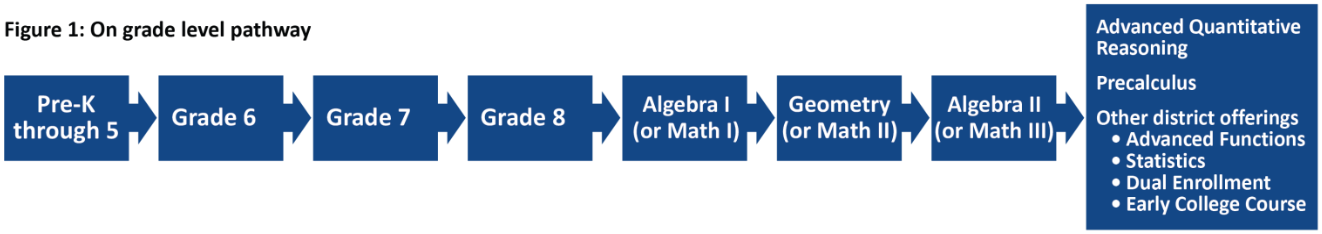


Figure 2–4: Three accelerated pathways leading to Calculus

Figure 2:

Compacting in Middle School

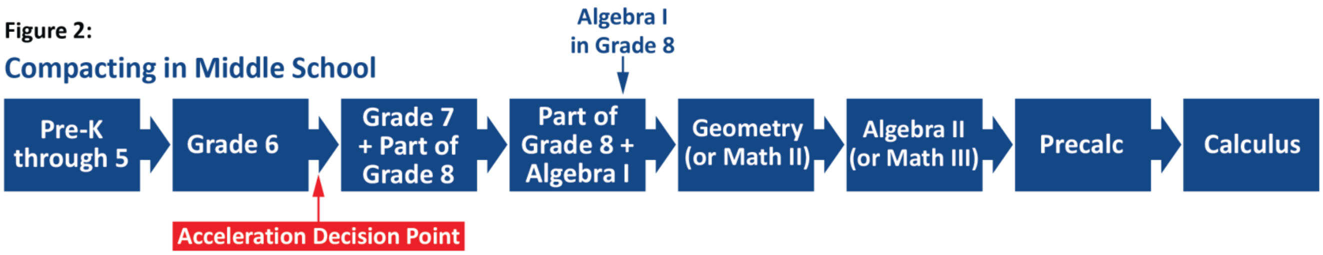
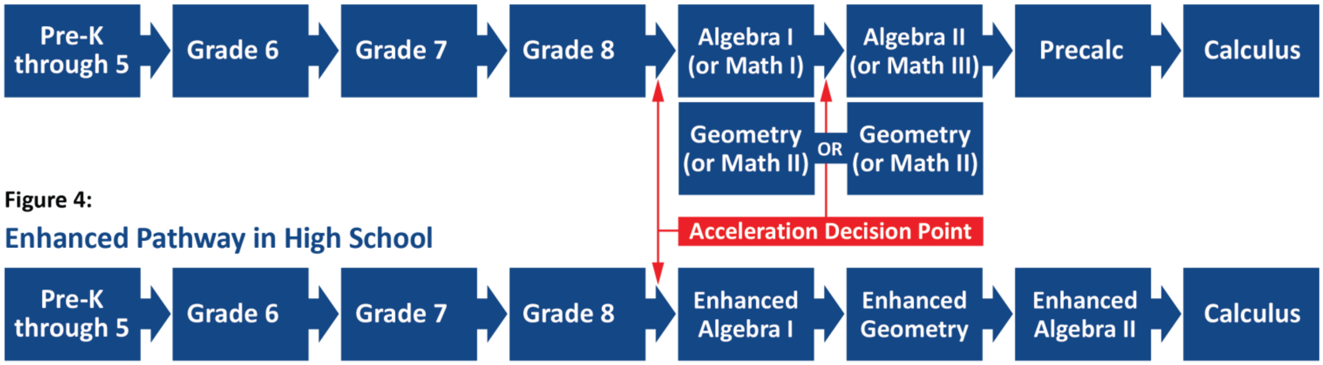


Figure 3:

Doubling Up in High School

Figure 4:

Enhanced Pathway in High School



Appendix I: Application of Standards for English Learners and Students with Disabilities

English Learners

The Massachusetts Department of Elementary and Secondary Education (ESE) strongly believes that all students, including English learners (ELs) should be held to the same high expectations outlined in the Curriculum Framework. English learners may require additional time and support as they work to acquire English language proficiency and content-area knowledge simultaneously. Further, developing proficiency in English takes time, and teachers should recognize that it is possible to meet the standards for mathematical content and practices as students become fluent in English.

The structure of programs serving ELs in Massachusetts acknowledges that ELs acquire language while interacting in all classrooms. All educators, including mathematics teachers, are responsible for students' language development and academic achievement. Collaboration and shared responsibility among administrators and educators are integral to student and program success. ESE uses the term *English language development* (ELD) to describe all of the language development that takes place throughout a student's day, both during sheltered content instruction (SCI) in math and in ESL classrooms. Together SCI and ESL comprise a complete program of sheltered English immersion (SEI).[30]

Districts in Massachusetts must provide EL students with both grade-level academic math content and ESL instruction that is aligned to the World Class Instructional Design and Assessment standards or WIDA and the Curriculum Frameworks as outlined in state guidelines for EL programs. ESE's Office of English Language Acquisition and Academic Achievement (OELAAA) offers a number of resources to help districts meet these expectations, including a Next-Generation ESL Curriculum Resource Guide, a set of ESL Model Curriculum Units with connections to ESE Model Curriculum Units (MCUs) in various content areas, and a Collaboration tool that supports WIDA standards implementation in conjunction with the Massachusetts Curriculum Frameworks. In partnership with educators, as well as other state and national experts, OELAAA is also developing a suite of updated SEI resources including comprehensive programmatic and curricular guidance for districts and eight new sheltered content immersion MCUs.

Regardless of the specific curriculum used, *all* ELs in formal educational settings must have access to:

- District and school personnel with the skills and qualifications necessary to support ELs' growth.
- Literacy-rich environments where students are immersed in a variety of robust language experiences.
- Speakers of English who know the language well enough to provide models and support.

Yet English learners are a heterogeneous group, with differences in cultural background, home language(s), socioeconomic status, educational experiences, and levels of English language proficiency. Educating ELs effectively requires diagnosing each student instructionally, tailoring instruction to individual needs, and monitoring progress closely and continuously. For example, ELs who are literate in a home language that shares cognates with English can apply home-language vocabulary knowledge when reading in English; likewise, those with extensive schooling can use conceptual knowledge developed in another language when learning academic content in English. Students with limited or interrupted formal schooling (SLIFE) may need to acquire more background knowledge before engaging in the educational task at hand.

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[30] For more on types of English Learner Education (ELE) programs in Massachusetts, please see Guidance on Identification, Assessment, Placement, and Reclassification of English Language Learners.

Six key principles should therefore guide instruction for ELs:[31]

- Instruction focuses on providing ELs with opportunities to engage in math-specific practices that build conceptual understanding and language competence in tandem.
- Instruction leverages ELs' home language(s), cultural assets, and prior math knowledge.
- Standards-aligned instruction for ELs is rigorous, grade-level appropriate, and provides deliberate, appropriate, and nuanced scaffolds.
- Instruction moves ELs forward by taking into account their English proficiency level(s) and prior schooling experiences.
- Instruction fosters ELs' autonomy by equipping them with the strategies necessary to comprehend and use language in mathematics classrooms.
- Responsive diagnostic tools and formative assessment practices measure ELs' mathematics content knowledge, language competence, and participation in mathematics practices.

In sum, the *Massachusetts Curriculum Framework for Mathematics* articulates rigorous grade-level expectations in the standards for mathematics content and mathematics practice to prepare all students, including ELs, for postsecondary education, careers, and everyday life. This document can be used in conjunction with language development standards designed to guide and monitor ELs' progress toward English proficiency. Many English learners also benefit from instruction on negotiating situations outside of schooling and career—instruction that enables them to participate on equal footing with English proficient peers in all aspects of social, economic, and civic life. Whether academic, linguistic, or social, support for ELs must be grounded in respect for the great value that multilingualism and multiculturalism add to our society.

Students with Disabilities

The *Massachusetts Curriculum Framework for Mathematics* articulates rigorous grade-level expectations. These learning standards identify the mathematical knowledge and skills all students need in order to be successful in college and careers and in everyday life. Students with disabilities—students eligible under the Individuals with Disabilities Education Act (IDEA)—must be challenged to excel within the general mathematics curriculum and be prepared for success in their post-school lives, including college and/or careers. The standards provide an opportunity to improve access to rigorous mathematics content for students with disabilities. The continued development of understanding about research-based instructional practices and a focus on their effective implementation will help improve access to the mathematics content standards and the mathematics practice standards for all students, including those with disabilities.

Students with disabilities are a heterogeneous group. Students who are eligible for an Individualized Education Program (IEP) have one or more disabilities and, as a result of the disability/ies, are unable to progress effectively in the general education program without the provision of specially designed instruction, or are unable to access the general mathematics curriculum without the provision of one or more related services (603 CMR 28.05 (2)(a)(1). *How* these high standards are taught and assessed is of importance in reaching students with diverse needs. In order for students with disabilities to meet high academic standards, their math instruction must incorporate individualized instruction or related services, supports, and accommodations necessary to allow the student to access the general mathematics curriculum. The annual goals included in students' IEPs must be carefully aligned to and facilitate students' attainment of grade-level learning standards.

Promoting a culture of high expectations for all students is a fundamental goal of the Massachusetts Curriculum Frameworks. In order to participate successfully in the general curriculum, students with disabilities may be provided additional supports and services as identified in their IEPs, including:

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[31] For more on the Six Key Principles for EL Instruction, please see Principles for ELL Instruction (2013, January). Understanding Language.

- Instructional learning supports based on the principles of Universal Design for Learning (UDL) which foster student engagement by presenting information in multiple ways and allowing for diverse avenues of demonstration, response, action, and expression. UDL is defined by the Higher Education Opportunity Act (PL 110-135) as "a scientifically valid framework for guiding educational practice that (a) provides flexibility in the ways information is presented, in the ways students respond or demonstrate knowledge and skills, and in the ways students are engaged; and (b) reduces barriers in instruction, provides appropriate accommodations, supports, and challenges, and maintains high achievement expectations for all students, including students with disabilities and students who are limited English proficient."
- Instructional accommodations (Thompson, Morse, Sharpe & Hall, 2005), such as alternative materials or procedures that do not change the standards or expectations, but allow students to learn within the framework of the general curriculum.
- Assistive technology devices and services to ensure access to the general education curriculum and the Massachusetts standards for mathematics.

Some students with the most significant cognitive disabilities will require substantial supports and accommodations to have meaningful access to certain standards in both instruction and assessment, based on their expressive communication and academic needs. These supports and accommodations must be identified in the students' IEPs and should ensure that students receive access to multiple means of learning, and opportunities to demonstrate knowledge, but at the same time retain the rigor and high expectations of the Mathematics Curriculum Framework.

References:

Individuals with Disabilities Education Act (IDEA), 34 CFR §300.34 (a). (2004).

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Thompson, Sandra J., Amanda B. Morse, Michael Sharpe, and Sharon Hall. "Accommodations Manual: How to Select, Administer and Evaluate Use of Accommodations and Assessment for Students with Disabilities," 2nd Edition. Council for Chief State School Officers, 2005 http://www.ccsso.org/content/pdfs/AccommodationsManual.pdf. (Accessed January 29, 2010).

Appendix II: Standards for Mathematical Practice Grade-Span Descriptions: Pre-K–5, 6–8, 9–12

Standards for Mathematical Practice Grades Pre-K–5

1. Make sense of problems and persevere in solving them.

Mathematically proficient elementary students explain to themselves the meaning of a problem, look for entry points to begin work on the problem, and plan and choose a solution pathway. For example, young students might use concrete objects or pictures to show the actions of a problem, such as counting out and joining two sets to solve an addition problem. If students are not at first making sense of a problem or seeing a way to begin, they ask questions that will help them get started. As they work, they continually ask themselves, "Does this make sense?" When they find that their solution pathway does not make sense, they look for another pathway that does. They may consider simpler forms of the original problem; for example, to solve a problem involving multi-digit numbers, they might first consider similar problems that involve multiples of ten or one hundred. Once they have a solution, they look back at the problem to determine if the solution is reasonable and accurate. They often check their answers to problems using a different method or approach. Mathematically proficient students consider different representations of the problem and different solution pathways, both their own and those of other students, in order to identify and analyze correspondences among approaches. They can explain correspondences among physical models, pictures, diagrams, equations, verbal descriptions, tables, and graphs.

2. Reason abstractly and quantitatively.

Mathematically proficient elementary students make sense of quantities and their relationships in problem situations. They can contextualize quantities and operations by using images or stories. They interpret symbols as having meaning, not just as directions to carry out a procedure. Even as they manipulate the symbols, they can pause as needed to access the meaning of the numbers, the units, and the operations that the symbols represent. Mathematically proficient students know and flexibly use different properties of operations, numbers, and geometric objects. They can contextualize an abstract problem by placing it in a context they then use to make sense of the mathematical ideas. For example, when a student sees the expression 40-26, the student might visualize this problem by thinking, if I have 26 marbles and Marie has 40, how many more do I need to have as many as Marie? Then, in that context, the student thinks, 4 more will get me to a total of 30, and then 10 more will get me to 40, so the answer is 14. In this example, the student uses a context to think through a strategy for solving the problem, using the relationship between addition and subtraction and decomposing and recomposing the quantities. The student then uses what he/she did in the context to identify the solution of the original abstract problem. Mathematically proficient students can also make sense of a contextual problem and express the actions or events that are described in the problem using numbers and symbols. If they work with the symbols to solve the problem, they can then interpret their solution in terms of the context.

3. Construct viable arguments and critique the reasoning of others.

Mathematically proficient elementary students construct verbal and written mathematical arguments—that is, explain the reasoning underlying a strategy, solution, or conjecture—using concrete referents such as objects, drawings, diagrams, and actions. Arguments may also rely on definitions, previously established results, properties, or structures. For example, a student might argue that two different shapes have equal area because it has already been demonstrated that both shapes are half of the same rectangle. Students might also use counterexamples to argue that a conjecture is not true—for example, a rhombus is an example that shows that not all quadrilaterals with 4 equal sides are squares; or, multiplying by 1 shows that a product of two whole numbers is not always greater than each factor. Mathematically proficient students present their arguments in the form of representations, actions on those representations, and explanations in words (oral or written). In the elementary grades, arguments are often a combination of all three. Some of their arguments apply to individual problems, but others are about conjectures based on regularities they have noticed across multiple problems (see MP.8). As they articulate and justify generalizations, students consider to which mathematical objects (numbers or shapes, for example) their generalizations apply. For example, young students may believe a generalization about the behavior of addition applies to positive whole numbers less than 100 because those are the numbers with which they are currently familiar. As they expand their understanding of the number system, they may reexamine their conjecture for numbers in the hundreds and thousands. In upper elementary grades, students return to their conjectures and arguments about whole numbers to determine whether they apply to fractions and decimals. Mathematically proficient students can listen to or read the arguments of others, decide whether they make sense, ask useful questions to clarify or improve the arguments, and build on those arguments. They can communicate their arguments both orally and in writing, compare them to others, and reconsider their own arguments in response to the critiques of others.

4. Model with mathematics.

When given a problem in a contextual situation, mathematically proficient elementary students can identify the mathematical elements of a situation and create or interpret a mathematical model that shows those elements and relationships among them. The mathematical model might be represented in one or more of the following ways: numbers and symbols; geometric figures, pictures, or physical objects used to abstract the mathematical elements of the situation; a mathematical diagram such as a number line, table, or graph; or students might use more than one of these to help them interpret the situation. For example, when students encounter situations such as sharing a pan of cornbread among six people, they might first show how to divide the cornbread into six equal pieces using a picture of a rectangle. The rectangle divided into 6 equal pieces is a model of the essential mathematical elements of the situation. When the students learn to write the name of each piece in relation to the whole pan as ^{1} experiments, and observational $_6$, they are now modeling the situation with mathematical notation. Mathematically proficient students are able to identify important quantities in a contextual situation and use mathematical models to show the relationships of those quantities, particularly in multi-step problems or problems involving more than one variable. For example, if there is a penny jar that starts with three pennies in the jar, and four pennies are added each day, students might use a table to model the relationship between number of days and number of pennies in the jar. They can then use the model to determine how many pennies are in the jar after 10 days, which in turn helps them model the situation with the expression, 4 x 10 + 3. Mathematically proficient students use and interpret models to analyze relationships and draw conclusions. They interpret their mathematical results in the context of the situation and reflect on whether the results make sense, possibly improving the model if it has not served its purpose. As students model situations with mathematics, they are choosing tools appropriately (MP.5). As they decontextualize the situation and represent it mathematically, they are also reasoning abstractly (MP.2).

5. Use appropriate tools strategically.

Mathematically proficient elementary students consider the tools that are available when solving a mathematical problem, whether in a real-world or mathematical context. These tools might include physical objects (cubes, geometric shapes, place value manipulatives, fraction bars, etc.); drawings or diagrams (number lines, tally marks, tape diagrams, arrays, tables, graphs, etc.); models of mathematical concepts, paper and pencil, rulers and other measuring tools, scissors, tracing paper, grid paper, virtual manipulatives, appropriate software applications, or other available technologies. Examples: a student may use graph paper to find all the possible rectangles that have a given perimeter or use linking cubes to represent two quantities and then compare the two representations side by side. Proficient students are sufficiently familiar with tools appropriate for their grade and areas of content to make sound decisions about when each of these tools might be helpful, recognizing both the insight to be gained from their use as well as their limitations. Students choose tools that are relevant and useful to the problem at hand. These include tools mentioned above, as well as mathematical tools such as estimation or a particular strategy or algorithm. For example, in order to solve $^{3}/_{5}$ – ½, a student might recognize that knowledge of equivalents of ½ is an appropriate tool: since ½ is equivalent to 2½ fifths, the result is ½ of a fifth or $^{1}/_{10}$. This practice is also related to looking for structure (MP.7), which often results in building mathematical tools that can then be used to solve problems.

6. Attend to precision.

Mathematically proficient elementary students communicate precisely to others both verbally and in writing. They start by using everyday language to express their mathematical ideas, realizing that they need to select words with clarity and specificity rather than saying, for example, “it works" without explaining what “it" means. As they encounter the ambiguity of everyday terms, they come to appreciate, understand, and use mathematical vocabulary. Once young students become familiar with a mathematical idea or object, they are ready to learn more precise mathematical terms to describe it. In using mathematical representations, students use care in providing appropriate labels to precisely communicate the meaning of their representations. When making mathematical arguments about a solution, strategy, or conjecture (see MP.3), mathematically proficient students learn to craft careful explanations that communicate their reasoning by referring specifically to each important mathematical element, describing the relationships among them, and connecting their words clearly to their representations. Elementary students use mathematical symbols correctly and can describe the meaning of the symbols they use. When measuring, mathematically proficient students use tools and strategies to minimize the introduction of error. Mathematically proficient students specify units of measure; label charts, graphs, and drawings; calculate accurately and efficiently; and use clear and concise notation to record their work.

7. Look for and make use of structure.

Mathematically proficient elementary students use structures such as place value, the properties of operations, other generalizations about the behavior of the operations (for example, even numbers can be divided into 2 equal groups and odd numbers, when divided by 2, always have 1 left over), and attributes of shapes to solve problems. In many cases, they have identified and described these structures through repeated reasoning (MP.8). For example, when younger students recognize that adding 1 results in the next counting number, they are identifying the basic structure of whole numbers. When older students calculate 16 x 9, they might apply the structure of place value and the distributive property to find the product: 16 x 9 = (10 + 6) x 9 = (10 x 9) + (6 x 9). To determine the volume of a 3 x 4 x 5 rectangular prism, students might see the structure of the prism as five layers of 3 x 4 arrays of cubes.

8. Look for and express regularity in repeated reasoning.

Mathematically proficient elementary students look for regularities as they solve multiple related problems, then identify and describe these regularities. For example, students might notice a pattern in the change to the product when a factor is increased by 1: 5 x 7 = 35 and 5 x 8 = 40—the product changes by 5; 9 x 4 = 36 and 10 x 4 = 40—the product changes by 4. Students might then express this regularity by saying something like, “When you change one factor by 1, the product increases by the other factor." Younger students might notice that when tossing two-color counters to find combinations of a given number, they always get what they call “opposites"—when tossing 6 counters, they get 2 red, 4 yellow and 4 red, 2 yellow and when tossing 4 counters, they get 1 red, 3 yellow and 3 red, 1 yellow. Mathematically proficient students formulate conjectures about what they notice, for example, that when 1 is added to a factor, the product increases by the other factor. As students practice articulating their observations both verbally and in writing, they learn to communicate with greater precision (MP.6). As they explain why these generalizations must be true, they construct, critique, and compare arguments (MP.3).

Standards for Mathematical Practice Grades 6–8

1. Make sense of problems and persevere in solving them.

Mathematically proficient middle school students set out to understand a problem and then look for entry points to its solution. They analyze problem conditions and goals, translating, for example, verbal descriptions into mathematical expressions, equations, or drawings as part of the process. They consider analogous problems, and try special cases and simpler forms of the original in order to gain insight into its solution. For example, to understand why a 20% discount followed by a 20% markup does not return an item to its original price, they might translate the situation into a tape diagram or a general equation; or they might first consider the result for an item priced at $1.00 or $10.00. Mathematically proficient students can explain how alternate representations of problem conditions relate to each other. For example, they can navigate among tables, graphs, and equations representing linear relationships to gain insights into the role played by constant rate of change. Mathematically proficient students check their answers to problems and they continually ask themselves, “Does this make sense?” and “Can I solve the problem in a different way?” While working on a problem, they monitor and evaluate their progress and change course if necessary. They can understand the approaches of others to solving complex problems and compare approaches.

2. Reason abstractly and quantitatively.

Mathematically proficient middle school students make sense of quantities and relationships in problem situations. For example, they can apply ratio reasoning to convert measurement units and proportional relationships to solve percent problems. They represent problem situations using symbols and then manipulate those symbols in search of a solution (decontextualize). They can, for example, solve problems involving unit rates by representing the situations in equation form. Mathematically proficient students also pause as needed during problem solving to double-check the meaning of the symbols involved. In the process, they can look back at the applicable units of measure to clarify or inform solution steps (contextualize). Students can integrate quantitative information and concepts expressed in text and visual formats. Quantitative reasoning also entails knowing and flexibly using different properties of operations and objects. For example, in middle school, students use properties of operations to generate equivalent expressions and use the number line to understand multiplication and division of rational numbers.

3. Construct viable arguments and critique the reasoning of others.

Mathematically proficient middle school students understand and use assumptions, definitions, and previously established results in constructing verbal and written arguments. They make and explore the validity of conjectures. They can recognize and appreciate the use of counterexamples, for example, using numerical counterexamples to identify common errors in algebraic manipulation, such as thinking that $5 - 2x$ is equivalent to $3x$. Mathematically proficient students can explain and justify their conclusions using numerals, symbols, and visuals, communicate them to others, and respond to the arguments of others. They reason inductively about data, making plausible arguments that take into account the context from which the data arose. For example, they might argue that the great variability of heights in their class is explained by growth spurts, and that the small variability of ages is explained by school admission policies. Mathematically proficient students are also able to compare the effectiveness of two plausible arguments, distinguish correct logic or reasoning from that which is flawed, and if there is a flaw in an argument, explain what it is. Students engage in collaborative discussions, drawing on evidence from texts and arguments of others, follow conventions for collegial discussions, and qualify their own views in light of evidence presented. They consider questions such as “How did you get that?” “Why is that true?” and “Does that always work?”

4. Model with mathematics.

Mathematically proficient middle school students can apply the mathematics they know to solve problems arising in everyday life, society, and the workplace. This might be as simple as translating a verbal or written description to a drawing or mathematical expression. It might also entail applying proportional reasoning to plan a school event or using a set of linear inequalities to analyze a problem in the community. Mathematically proficient students are comfortable making assumptions and approximations to simplify a complicated situation, realizing that these may need revision later. For example, they can roughly fit a line to a scatter plot to make predictions and gather experimental data to approximate a probability. They are able to identify important quantities in a given relationship such as rates of change and represent situations using such tools as diagrams, tables, graphs, flowcharts and formulas. They can analyze their representations mathematically, use the results in the context of the situation, and then reflect on whether the results make sense while possibly improving the model.

5. Use appropriate tools strategically.

Mathematically proficient middle school students strategically consider the available tools when solving a mathematical problem and while exploring a mathematical relationship. These tools might include pencil and paper, concrete models, a ruler, a protractor, a graphing calculator, a spreadsheet, a statistical package, or dynamic geometry software. Proficient students make sound decisions about when each of these tools might be helpful, recognizing both the insights to be gained and their limitations. For example, they use estimation to check reasonableness, graphs to model functions, algebra tiles to see how properties of operations apply to algebraic expressions, graphing calculators to solve systems of equations, and dynamic geometry software to discover properties of parallelograms. When making mathematical models, they know that technology can enable them to visualize the results of their assumptions, to explore consequences, and to compare predictions with data. Mathematically proficient students are able to identify relevant external mathematical resources, such as digital content located on a website, and use them to pose or solve problems.

6. Attend to precision.

Mathematically proficient middle school students communicate precisely to others both verbally and in writing. They present claims and findings, emphasizing salient points in a focused, coherent manner with relevant evidence, sound and valid reasoning, well-chosen details, and precise language. They use clear definitions in discussion with others and in their own reasoning and determine the meaning of symbols, terms, and phrases as used in specific mathematical contexts. For example, they can use the definition of rational numbers to explain why a number is irrational and describe congruence and similarity in terms of transformations in the plane. They state the meaning of the symbols they choose, consistently and appropriately, such as inputs and outputs represented by function notation. They are careful about specifying units of measure, and label axes to display the correct correspondence between quantities in a problem. They calculate accurately and efficiently, express numerical answers with a degree of precision appropriate to the context. For example, they accurately apply scientific notation to large numbers and use measures of center to describe data sets.

7. Look for and make use of structure.

Mathematically proficient middle school students look closely to discern a pattern or structure. They might use the structure of the number line to demonstrate that the distance between two rational numbers is the absolute value of their difference, ascertain the relationship between slopes and solution sets of systems of linear equations, and see that the equation $3x = 2y$ represents a proportional relationship with a unit rate of $3/2 = 1.5$. They might recognize how the Pythagorean Theorem is used to find distances between points in the coordinate plane and identify right triangles that can be used to find the length of a diagonal in a rectangular prism. They also can step back for an overview and shift perspective, as in finding a representation of consecutive numbers that shows all sums of three consecutive whole numbers are divisible by six. They can see complicated things as single objects, such as seeing two successive reflections across parallel lines as a translation along a line perpendicular to the parallel lines or understanding 1.05*a* as an original value, *a*, plus 5% of that value, 0.05*a*.

8. Look for and express regularity in repeated reasoning.

Mathematically proficient middle school students notice if calculations are repeated, and look for both general methods and shortcuts. Working with tables of equivalent ratios, they might deduce the corresponding multiplicative relationships and make generalizations about the relationship to rates. By paying attention to the calculation of slope as they repeatedly check whether points are on the line through (1; 2) with slope 3, students might abstract the equation $^{(y-2)}/_{(x-1)} = 3$. Noticing the regularity with which interior angle sums increase with the number of sides in a polygon might lead them to the general formula for the interior angle sum of an *n*-gon. As they work to solve a problem, mathematically proficient students maintain oversight of the process while attending to the details. They continually evaluate the reasonableness of their intermediate results.

Standards for Mathematical Practice Grades 9–12

1. Make sense of problems and persevere in solving them.

Mathematically proficient high school students analyze givens, constraints, relationships, and goals. They make conjectures about the form and meaning of the solution and plan a solution pathway rather than simply jumping into a solution attempt. They consider analogous problems, and try special cases and simpler forms of the original problem in order to gain insight into its solution. They monitor and evaluate their progress and change course if necessary. High school students might, depending on the context of the problem, transform algebraic expressions or change the viewing window on their graphing calculator to get the information they need. Mathematically proficient students can explain correspondences between equations, verbal descriptions, tables, and graphs or draw diagrams of important features and relationships, graph and interpret representations of data, and search for regularity or trends. Mathematically proficient students check their answers to problems using different methods of solving, and they continually ask themselves, “Does this make sense?” They can understand the approaches of others to solving complex problems and identify correspondences between different approaches.

2. Reason abstractly and quantitatively.

Mathematically proficient high school students make sense of the quantities and their relationships in problem situations. Students bring two complementary abilities to bear on problems involving quantitative relationships: the ability to decontextualize—to abstract a given situation and represent it symbolically, and manipulate the representing symbols as if they have a life of their own, without necessarily attending to their referents—and the ability to contextualize, to pause as needed during the manipulation process in order to probe into the referents for the symbols involved. Students can write explanatory text that conveys their mathematical analyses and thinking, using relevant and sufficient facts, concrete details, quotations, and coherent development of ideas. Students can evaluate multiple sources of information presented in diverse formats (and media) to address a question or solve a problem. Quantitative reasoning entails habits of creating a coherent representation of the problem at hand; considering the units involved; attending to the meanings of quantities, not just how to compute them; and knowing and flexibly using different properties of operations and objects.

3. Construct viable arguments and critique the reasoning of others.

Mathematically proficient high school students understand and use stated assumptions, definitions, and previously established results in constructing verbal and written arguments. They make conjectures and build a logical progression of statements to explore the truth of their conjectures. They are able to analyze situations by breaking them into cases, and can recognize and use counterexamples and specific textual evidence to form their arguments. They justify their conclusions, communicate them to others, and respond to the arguments of others. They reason inductively about data, making plausible arguments that take into account the context from which the data arose. Mathematically proficient students are also able to compare the effectiveness of two plausible arguments, distinguish correct logic or reasoning from that which is flawed, and—if there is a flaw in an argument—explain what it is and why. They can construct formal arguments relevant to specific contexts and tasks. High school students learn to determine domains to which an argument applies. Students listen or read the arguments of others, decide whether they make sense, and ask useful questions to clarify or improve the arguments. Students engage in collaborative discussions, respond thoughtfully to diverse perspectives and approaches, and qualify their own views in light of evidence presented.

4. Model with mathematics.

Mathematically proficient high school students can apply the mathematics they know to solve problems arising in everyday life, society, and the workplace. By high school, a student might use geometry to solve a design problem or use a function to describe how one quantity depends on another. Mathematically proficient students who can apply what they know are comfortable making assumptions and approximations to simplify a complicated situation, realizing that these may need revision later. They are able to identify important quantities in a practical situation and map their relationships using such tools as diagrams, two-way tables, graphs, flowcharts and formulas. They can analyze those relationships mathematically to draw conclusions. They routinely interpret their mathematical results in the context of the situation and reflect on whether the results make sense, possibly improving the model if it has not served its purpose.

5. Use appropriate tools strategically.

Mathematically proficient high school students consider the available tools when solving a mathematical problem. These tools might include pencil and paper, concrete models, a ruler, a protractor, a calculator, a spreadsheet, a computer algebra system, a statistical package, or dynamic geometry software. Proficient students are sufficiently familiar with tools appropriate for high school to make sound decisions about when each of these tools might be helpful, recognizing both the insight to be gained and their limitations. For example, mathematically proficient high school students analyze graphs of functions and solutions generated using a graphing calculator. They detect possible errors by strategically using estimation and other mathematical knowledge. They are able to use technological tools to explore and deepen their understanding of concepts. When making mathematical models, they know that technology can enable them to visualize the results of varying assumptions, explore consequences, and compare predictions with data. Mathematically proficient students are able to identify relevant external mathematical resources, such as digital content located on a website, and use them to pose or solve problems.

6. Attend to precision.

Mathematically proficient high school students communicate precisely to others both verbally and in writing, adapting their communication to specific contexts, audiences, and purposes. They develop the habit of using precise language, not only as a mechanism for effective communication, but also as a tool for understanding and solving problems. Describing their ideas precisely helps students understand the ideas in new ways. They use clear definitions in discussions with others and in their own reasoning. They state the meaning of the symbols that they choose. They are careful about specifying units of measure, labeling axes, defining terms and variables, and calculating accurately and efficiently with a degree of precision appropriate for the problem context. They develop logical claims and counterclaims fairly and thoroughly in a way that anticipates the audiences' knowledge, concerns, and possible biases. High school students draw specific evidence from informational sources to support analysis, reflection, and research. They critically evaluate the claims, evidence and reasoning of others and attend to important distinctions with their own claims or inconsistencies in competing claims. Students evaluate the conjectures and claims, data, analysis, and conclusions in texts that include quantitative elements, comparing those with information found in other sources.

7. Look for and make use of structure.

Mathematically proficient high school students look closely to discern a pattern or structure. In the expression $x^2 + 9x + 14$, high school students can see the 14 as 2×7 and the 9 as $2 + 7$. They recognize the significance of an existing line in a geometric figure and can use the strategy of drawing an auxiliary line for solving problems. They also can step back for an overview and shift perspective. They can see complicated things, such as some algebraic expressions, as single objects or as being composed of several objects. For example, they can see $5 - 3(x - y)^2$ as 5 minus a positive number times a square, and use that to realize that its value cannot be more than 5 for any real numbers x and y.

8. Look for and express regularity in repeated reasoning.

Mathematically proficient high school students notice if calculations are repeated, and look both for general methods and for shortcuts. Noticing the regularity in the way terms sum to zero when expanding $(x - 1)(x + 1)$, $(x - 1)(x^2 + x + 1)$, and $(x - 1)(x^3 + x^2 + x + 1)$ might lead students to the general formula for the sum of a geometric series. As they work to solve a problem, mathematically proficient students maintain oversight of the process, while attending to the details and continually evaluating the reasonableness of their intermediate results.

Appendix III: High School Conceptual Category Tables

The mathematical content standards were designed for students to attain mathematical skills and concepts in a progression over time and across grade spans. The progressions were informed by research and by the logic of the mathematics. Conceptual categories are groups of inter-related standards. The tables describe how these conceptual category content standards are distributed across the model courses.

Distribution of Content Standards by five Conceptual Categories:

- **Number and Quantity (N)**
- **Algebra (A)**
- **Functions (F)**
- **Statistics and Probability (S)**
- **Geometry (G)**

Each Conceptual Category table shows the distribution of standards across the eight Model Courses:

- **Algebra I (AI)**
- **Geometry (GEO)**
- **Algebra II (AII)**
- **Math I (MI)**
- **Math II (MII)**
- **Math III (MIII)**
- **Precalculus (PC)**
- **Advanced Quantitative Reasoning (AQR)**

Number and Quantity [N]

	A I	GEO	A II	M I	M II	M III	PC	AQR
The Real Number System (N-RN)								
A. Extend the properties of exponents to rational exponents.								
1	✓				✓			
2	✓				✓			
B. Use properties of rational and irrational numbers.								
3	✓				✓			
Quantities (N-Q)								
A. Reason quantitatively and use units to solve problems.								
1	✓			✓				
2	✓			✓				
3	✓	✓		✓	✓			
a		✓			✓			
The Complex Number System (N-CN)								
A. Perform arithmetic operations with complex numbers.								
1			✓		✓			
2			✓		✓			
3+							✓	
B. Represent complex numbers and their operations on the complex plane.								
4+							✓	
5+							✓	
6+							✓	
C. Use complex numbers in polynomial identities and equations.								
7			✓		✓			
8+			✓			✓	✓	
9+			✓			✓	✓	
Vector and Matrix Quantities (N-VM)								
A. Represent and model with vector quantities.								
1+			✓			✓	✓	✓
2+							✓	✓
3+			✓			✓	✓	✓
B. Perform operations on vectors.								
4+							✓	
a+							✓	
b+							✓	
c+							✓	

	A I	GEO	A II	M I	M II	M III	PC	AQR
5+							✓	
a+							✓	
b+							✓	
C. Perform operations on matrices and use matrices in applications.								
6+			✓			✓	✓	✓
7+							✓	✓
8+			✓			✓	✓	✓
9+							✓	✓
10+							✓	✓
11+							✓	✓
12+			✓			✓	✓	✓

Algebra [A]

	A I	GEO	A II	M I	M II	M III	PC	AQR
Seeing Structure in Expressions (A-SSE)								
A. Interpret the structure of expressions.								
1	✓		✓	✓	✓	✓		
a	✓		✓	✓	✓	✓		
b	✓		✓	✓	✓	✓		
2	✓		✓		✓	✓		
B. Write expressions in equivalent forms to solve problems.								
3	✓				✓			
a	✓				✓			
b	✓				✓			
c	✓				✓			
4			✓			✓		
Arithmetic with Polynomials and Rational Expressions (A-APR)								
A. Perform arithmetic operations on polynomials.								
1	✓		✓		✓	✓		
a	✓		✓		✓	✓		
b	✓				✓			
B. Understand the relationship between zeros and factors of polynomials.								
2			✓			✓		
3			✓			✓		
C. Use polynomial identities to solve problems.								
4			✓			✓		
5+			✓			✓	✓	✓
D. Rewrite rational expressions.								
6			✓			✓		
7+			✓			✓	✓	
Creating Equations (A-CED)								
A. Create equations that describe numbers or relationships.								
1	✓		✓	✓	✓	✓		
2	✓		✓	✓	✓	✓		
3	✓		✓	✓		✓		
4	✓			✓	✓			
Reasoning with Equations and Inequalities (A-REI)								
A. Understand solving equations as a process of reasoning and explain the reasoning.								
1	✓			✓				
2			✓			✓		

	A I	GEO	A II	M I	M II	M III	PC	AQR
B. Solve equations and inequalities in one variable.								
3 **a**	✓ ✓			✓ ✓				
4 **a** **b**	✓ ✓ ✓				✓ ✓ ✓			
C. Solve systems of equations.								
5	✓			✓				
6	✓			✓				
7	✓				✓			
8+							✓	✓
9+							✓	✓
D. Represent and solve equations and inequalities graphically.								
10	✓			✓				
11	✓		✓	✓		✓		
12	✓			✓				

Functions [F]

	A I	GEO	A II	M I	M II	M III	PC	AQR
Interpreting Functions (F-IF)								
A. Understand the concept of a function and use function notation.								
1	✓			✓				
2	✓			✓				
3	✓			✓				
B. Interpret functions that arise in applications in terms of the context (linear, quadratic, exponential, rational, polynomial, square root, cube root, trigonometric, logarithmic).								
4	✓		✓	✓	✓	✓		
5	✓		✓	✓	✓	✓		
6	✓		✓	✓	✓	✓		
C. Analyze functions using different representations.								
7	✓		✓	✓	✓	✓	✓	
a	✓			✓	✓			
b	✓		✓		✓	✓		
c			✓			✓		
d+							✓	
e	✓		✓	✓		✓		
8	✓		✓		✓	✓		
a	✓		✓		✓	✓		
b	✓				✓	✓		
9	✓		✓	✓	✓	✓		
10			✓			✓		
Building Functions (F-BF)								
A. Build a function that models a relationship between two quantities.								
1	✓		✓	✓	✓	✓	✓	
a	✓			✓	✓			
b	✓		✓	✓	✓	✓		
c+							✓	
2	✓			✓				
B. Build new functions from existing functions.								
3	✓		✓	✓	✓	✓		
4	✓		✓		✓	✓	✓	
a	✓		✓		✓	✓		
b+							✓	
c+							✓	
d+							✓	
5+							✓	
Linear, Quadratic, and Exponential Models (F-LE)								
A. Construct and compare linear, quadratic, and exponential models and solve problems.								
1	✓			✓				
a	✓			✓				
b	✓			✓				
c	✓			✓				
2	✓			✓				
3	✓			✓	✓			
4			✓			✓		

	A I	GEO	A II	M I	M II	M III	PC	AQR
B. Interpret expressions for functions in terms of the situation they model.								
5	✓			✓				
Trigonometric Functions (F-TF)								
A. Extend the domain of trigonometric functions using the unit circle.								
1			✓			✓		
2			✓			✓		
3+							✓	✓
4+							✓	✓
B. Model periodic phenomena with trigonometric functions.								
5			✓			✓		
6+							✓	
7+							✓	✓
C. Prove and apply trigonometric identities.								
8			✓			✓		
9+							✓	✓

Statistics and Probability [S]

	A I	GEO	A II	M I	M II	M III	PC	AQR
Interpreting Categorical and Quantitative Data (S-ID)								
A. Summarize, represent, and interpret data on a single count or measurement variable. Use calculators, spreadsheets, and other technology as appropriate.								
1	✓			✓				
2	✓			✓				
3	✓			✓				
4	✓		✓			✓		
B. Summarize, represent, and interpret data on two categorical and quantitative variables.								
5	✓			✓				
6	✓			✓				
a	✓			✓				
b	✓			✓				
c	✓			✓				
C. Interpret linear models.								
7	✓			✓				
8	✓			✓				
9	✓			✓				✓
Making Inferences and Justifying Conclusions (S-IC)								
A. Understand and evaluate random processes underlying statistical experiments.								
1			✓			✓		
2			✓			✓		
B. Make inferences and justify conclusions from sample surveys, experiments, and observational studies.								
3			✓			✓		
4			✓			✓		✓
5			✓			✓		✓
6			✓			✓		✓
Conditional Probability and the Rules of Probability (S-CP)								
A. Understand independence and conditional probability and use them to interpret data.								
1		✓			✓			
2		✓			✓			
3		✓			✓			
4		✓			✓			
5		✓			✓			
B. Use the rules of probability to compute probabilities of compound events in a uniform probability model.								
6		✓			✓			
7		✓			✓			
8+		✓			✓			✓
9+		✓			✓			✓

	A I	GEO	A II	M I	M II	M III	PC	AQR
Using Probability to Make Decisions (S-MD)								
A. Calculate expected values and use them to solve problems.								
1+								✓
2+								✓
3+								✓
4+								✓
B. Use probability to evaluate outcomes of decisions.								
5+								✓
a+								✓
b+								✓
6+		✓	✓		✓	✓		✓
7+		✓	✓		✓	✓		✓

Geometry [G]

	A I	GEO	A II	M I	M II	M III	PC	AQR
Congruence (G-CO)								
A. Experiment with transformations in the plane.								
1		✓		✓				
2		✓		✓				
3		✓		✓				
4		✓		✓				
5		✓		✓				
B. Understand congruence in terms of rigid motions.								
6		✓		✓				
7		✓		✓				
8		✓		✓				
C. Prove geometric theorems.								
9		✓			✓			
10		✓			✓			
11		✓			✓			
a		✓			✓			
D. Make geometric constructions.								
12		✓		✓				
13		✓		✓				
Similarity, Right Triangles, and Trigonometry (G-SRT)								
A. Understand similarity in terms of similarity transformations.								
1		✓			✓			
a		✓			✓			
b		✓			✓			
2		✓			✓			
3		✓			✓			
B. Prove theorems involving similarity.								
4		✓			✓			
5		✓			✓			
C. Define trigonometric ratios and solve problems involving right triangles.								
6		✓			✓			
7		✓			✓			
8		✓			✓			
D. Apply trigonometry to general triangles.								
9+		✓				✓	✓	
10+		✓				✓	✓	
11+		✓				✓	✓	✓

	A I	GEO	A II	M I	M II	M III	PC	AQR
Circles (G-C)								
A. Understand and apply theorems about circles.								
1		✓			✓			
2		✓			✓			
3		✓			✓			
4+		✓			✓		✓	✓
B. Find arc lengths and areas of sectors of circles.								
5		✓			✓			
Expressing Geometric Properties with Equations (G-GPE)								
A. Translate between the geometric description and the equation for a conic section.								
1		✓			✓			
2		✓			✓			
3+							✓	✓
a+							✓	✓
B. Use coordinates to prove simple geometric theorems algebraically.								
4		✓			✓			
5		✓		✓				
6		✓			✓			
7		✓		✓				
Geometric Measurement and Dimension (G-GMD)								
A. Explain volume formulas and use them to solve problems.								
1		✓			✓			
2+		✓			✓		✓	✓
3		✓			✓			
B. Visualize relationships between two-dimensional and three-dimensional objects.								
4		✓			✓	✓		
Modeling with Geometry (G-MG)								
A. Apply geometric concepts in modeling situations.								
1		✓				✓		
2		✓				✓		
3		✓				✓		
4		✓				✓		

Glossary: Mathematical Terms, Tables, and Illustrations

This Glossary contains terms found in the 2017 *Massachusetts Curriculum Framework for Mathematics,* as well as selected additional terms.

Glossary Sources
(H) Harcourt School Publishers Math Glossary
(M) Merriam-Webster Dictionary
(MW) MathWords.com
(NCTM) National Council of Teachers of Mathematics
(OK) Oklahoma State Department of Education

A

AA similarity. Angle-angle similarity. When two triangles have corresponding angles that are congruent, the triangles are similar. (MW)

ASA congruence. Angle-side-angle congruence. When two triangles have corresponding angles and sides that are congruent, the triangles themselves are congruent. (MW)

Absolute value. The absolute value of a real number is its (non-negative) distance from 0 on a number line.

Addition and subtraction within 5, 10, 20, 100, or 1,000. Addition or subtraction of two whole numbers with whole number answers, and with sum or minuend in the range 0–5, 0–10, 0–20, or 0–100, respectively. *Example: 8 + 2 = 10 is an addition within 10, 14 – 5 = 9 is a subtraction within 20, and 55 – 18 = 37 is a subtraction within 100.*

Additive inverses. Two numbers whose sum is 0 are additive inverses of one another. *Example: 3/4 and –3/4 are additive inverses of one another because 3/4 + (–3/4) = (–3/4) + 3/4 = 0.*

Algorithm/Standard Algorithm:

Algorithm. A finite set of steps for completing a procedure, e.g., multi-digit operations (addition, subtraction, multiplication, division). *(See standard 3.NBT.A.2.)*

> **Standard algorithm.** One of the conventional algorithms used in the United States based on place value and properties of operations for addition, subtraction, multiplication, and division. *(See standards 4.NBT.B.4, 5.NBT.B.5, and 6.NS.B.2)*. See Table 5 in the Glossary.

Analog. Having to do with data represented by continuous variables, e.g., a clock with hour, minute, and second hands. (M)

Analytic geometry. The branch of mathematics that uses functions and relations to study geometric phenomena, e.g., the description of ellipses and other conic sections in the coordinate plane by quadratic equations.

Argument of a complex number. The angle describing the direction of a complex number on the complex plane. The argument is measured in radians as an angle in standard position. For a complex number in polar form $r(\cos \theta + i \sin \theta)$, the argument is θ. (MW)

Associative property of addition. See Table 3 in the Glossary.

Associative property of multiplication. See Table 3 in the Glossary.

Assumption. A fact or statement (as a proposition, axiom, postulate, or notion) taken for granted. (M)

Attribute. A common feature of a set of figures.

B

Benchmark fraction. A common fraction against which other fractions can be measured, such as ½.

Binomial Theorem. A theorem that gives the polynomial expansion for any whole-number power of a binomial. For powers greater than or equal to zero. (OK)

Bivariate data. Pairs of linked numerical observations. *Example: a list of heights and weights for each player on a football team.*

Box plot. A graphic method that shows the distribution of data values by using the median, quartiles, and extremes of the data set. A box shows the middle 50% of the data. (DPI)

C

Calculus. The mathematics of change and motion. The main concepts of calculus are limits, instantaneous rates of change, and areas enclosed by curves.

Capacity. The maximum amount or number that can be contained or accommodated, e.g., a jug with a one-gallon *capacity;* the auditorium was filled to *capacity.*

Cardinal number. A number (such as 1, 5, 15) that is used in simple counting and that indicates how many elements there are in a set.

Cartesian plane. A coordinate plane with perpendicular coordinate axes.

Cavalieri's Principle. A method, with formula given below, of finding the volume of any solid for which cross-sections by parallel planes have equal areas. This includes, but is not limited to, cylinders and prisms. Formula: Volume = Bh, where B is the area of a cross-section and h is the height of the solid. (MW)

Coefficient. Any of the factors of a product considered in relation to a specific factor. (W)

Commutative property. See Table 3 in the Glossary.

Compare two treatments. Compare different levels of a variable, imposed as treatments in an experiment, to each other and/or to a control group.

Complex fraction. A fraction A/B where A and/or B are fractions (B nonzero).

Complex number. A number that can be written as the sum or difference of a real number and an imaginary number. See Illustration 1 in the Glossary. (MW)

Complex plane. The coordinate plane used to graph complex numbers. (MW)

Compose numbers. a) Given pairs, triples, etc. of numbers, identify sums or products that can be computed; b) Each place in the base-ten place value is composed of ten units of the place to the left, i.e., one hundred is composed of ten bundles of ten, one ten is composed of ten ones, etc.

Compose shapes. Join geometric shapes without overlaps to form new shapes.

Composite number. A whole number that has more than two factors. (H)

Computation algorithm. A set of predefined steps applicable to a class of problems that gives the correct result in every case when the steps are carried out correctly. *See also: **algorithm**; **computation strategy**.*

Computation strategy. Purposeful manipulations that may be chosen for specific problems, may not have a fixed order, and may be aimed at converting one problem into another. *See also: **computation algorithm**.*

Congruent. Two plane or solid figures are congruent if one can be obtained from the other by rigid motion (a sequence of rotations, reflections, and translations).

Conjugate. The result of writing the sum of two terms as a difference, or vice versa. *For example, the conjugate of* $x - 2$ *is* $x + 2$. (MW)

Coordinate plane. A plane in which a point is represented using two coordinates that determine the precise location of the point. In the Cartesian plane, two perpendicular number lines are used to determine the locations of points. In the polar coordinate plane, points are determined by their distance along a ray through that point and the origin, and the angle that ray makes with a pre- determined horizontal axis. (OK)

Cosine. A trigonometric function that for an acute angle is the ratio between a leg adjacent to the angle when the angle is considered part of a right triangle and the hypotenuse. (M)

Counting number. A number used in counting objects, i.e., a number from the set 1, 2, 3, 4, 5,…. See Illustration 1 in the Glossary.

Counting on. A strategy for finding the number of objects in a group without having to count every member of the group. *For example, if a stack of books is known to have eight books and three more books are added to the top, it is not necessary to count the stack all over again; one can find the total by counting on—pointing to the top book and saying "eight," following this with "nine, ten, eleven. There are eleven books now."*

D

Decimal expansion. Writing a rational number as a decimal.

Decimal fraction. A fraction (as $0.25 = {}^{25}/_{100}$ or $0.025 = {}^{25}/_{1000}$) or mixed number (as $3.025 = 3\ {}^{25}/_{1000}$) in which the denominator is a power of ten, usually expressed by the use of the decimal point. (M)

Decimal number. Any real number expressed in base ten notation, such as 2.673.

Decompose numbers. Given a number, identify pairs, triples, etc. of numbers that combine to form the given number using subtraction and division.

Decompose shapes. Given a geometric shape, identify geometric shapes that meet without overlap to form the given shape.

Differences between parameters. A difference of numerical characteristics of a population, including measures of center and/or spread.

Digit. a) Any of the Arabic numerals 1 to 9 and usually the symbol 0; b) One of the elements that combine to form numbers in a system other than the decimal system.

Digital. Having to do with data that is represented in the form of numerical digits; providing a read out in numerical digits, e.g., a digital watch.

Dilation. A transformation that moves each point along the ray through the point emanating from a fixed center, and multiplies distances from the center by a common scale factor.

Directrix. A parabola is the collection of all points in the plane that are the same distance from a fixed point, called the focus (F), as they are from a fixed line, called the directrix (D). (Lone Star College lonetsar.edu)

Discrete mathematics. The branch of mathematics that includes combinatorics, recursion, Boolean algebra, set theory, and graph theory.

Dot plot. *See:* ***line plot***.

E

Expanded form. A multi-digit number is expressed in expanded form when it is written as a sum of single-digit multiples of powers of ten. *For example, 643 = 600 + 40 + 3.*

Expected value. For a random variable, the weighted average of its possible values, with weights given by their respective probabilities.

Exponent. The number that indicates how many times the base is used as a factor, e.g., in $4^3 = 4 \times 4 \times 4 = 64$, the exponent is 3, indicating that 4 is repeated as a factor three times.

Exponential function. A function of the form $y = a \bullet b^x$ where $a > 0$ and either $0 < b < 1$ or $b > 1$. The variables do not have to be x and y. *For example,* $A = 3.2 \bullet (1.02)^t$ *is an exponential function.*

Expression. A mathematical phrase that combines operations, numbers, and/or variables (e.g., $3^2 \div a$). (H)

F

Fibonacci sequence. The sequence of numbers beginning with 1, 1, in which each number that follows is the sum of the previous two numbers, i.e., 1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89, 144….

First quartile. For a data set with median *M*, the first quartile is the median of the data values less than *M*. *Example: For the data set {1, 3, 6, 7, 10, 12, 14, 15, 22, 120}, the first quartile is 6.*[32] *See also:* ***median, third quartile, interquartile range****.*

Fluency. Fluency in the grades 1–6 standards is the ability *to carry out calculations* and apply *numerical algorithms* quickly and accurately. Fluency in each grade involves a mixture of knowing some answers from memory (instant recall), knowing some answers from patterns (e.g., "adding 0 yields the same number"), and knowing some answers from the use of other strategies. The development of fluency follows a specific progression in these grades that begins with conceptual understanding and eventually requires students to "know from memory their math facts," use various strategies to arrive at answers, and develop proficiency using the standard algorithm for each operation. *(See standards 1.OA.B.3, 2.OA.B.2, 3.OA.B.5, 3.OA.C.7 and 3.NBT.A.2, 4.NBT.B.4, 5.NBT.B.5, 6.NS.B.2 and 6.NS.B.3.)*

Fraction. A number expressible in the form a/b where a is a whole number and b is a positive whole number. (The word *fraction* in these standards always refers to a nonnegative number.) *See also:* ***rational number.***

Function. A mathematical relation for which each element of the domain corresponds to exactly one element of the range. (MW)

Function notation. A notation that describes a function. For a function ƒ, when x is a member of the domain, the symbol $f(x)$ denotes the corresponding member of the range (e.g., $f(x) = x + 3$).

Fundamental Theorem of Algebra. The theorem that establishes that, using complex numbers, all polynomials can be factored into a product of linear terms. A generalization of the theorem asserts that any polynomial of degree n has exactly n zeros, counting multiplicity. (MW)

G

Geometric sequence (progression). An ordered list of numbers that has a common ratio between consecutive terms, e.g., 2, 6, 18, 54…. (H)

H

Histogram. A type of bar graph used to display the distribution of measurement data across a continuous range.

I

Identity property of 0. See Table 3 in the Glossary.

Imaginary number. Complex numbers with no real terms, such as $5i$. See Illustration 1 in the Glossary. (M)

Independently combined probability models. Two probability models are said to be combined independently if the probability of each ordered pair in the combined model equals the product of the original probabilities of the two individual outcomes in the ordered pair.

Integer. All positive and negative whole numbers, including zero. (MW)

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[32] Many different methods for computing quartiles are in use. The method defined here is sometimes called the Moore and McCabe method. See Langford, E., "Quartiles in Elementary Statistics," *Journal of Statistics Education* Volume 14, Number 3 (2006).

Interquartile range. A measure of variation in a set of numerical data, the interquartile range is the distance between the first and third quartiles of the data set. Example: For the data set {1, 3, 6, 7, 10, 12, 14, 15, 22, 120}, the interquartile range is 15 – 6 = 9. *See also:* ***first quartile, third quartile****.*

Inverse function. A function obtained by expressing the dependent variable of one function as the independent variable of another; that is the inverse of $y = f(x)$ is $x = f^{-1}(y)$. (NCTM)

Irrational number. A number that cannot be expressed as a quotient of two integers, e.g., $\sqrt{2}$. It can be shown that a number is irrational if and only if it cannot be written as a repeating or terminating decimal.

K

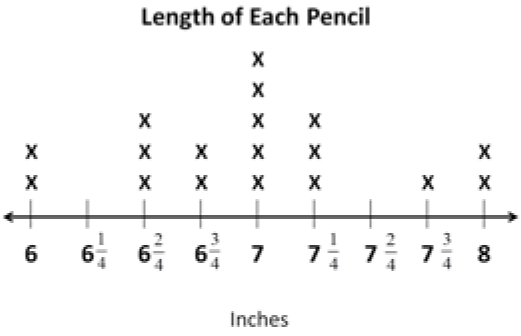
Know from Memory. To instantly recall single-digit math facts to use when needed. *Note: In the early grades, students develop number sense and fluency in operations. Students are expected to commit single digit math facts to memory by the end of: a) grade 2 for addition and related subtraction facts (see standard 2.OA.B.2); and b) grade 3 for multiplication and related division facts (see standard 3.OA.C.7).*

L

Law of Cosines. An equation relating the cosine of an interior angle and the lengths of the sides of a triangle. (MW)

Law of Sines. Equations relating the sines of the interior angles of a triangle and the corresponding opposite sides. (MW)

Line plot. A method of visually displaying a distribution of data values where each data value is shown as a dot or mark above a number line. (Also known as a *dot plot*.) (DPI)



Linear association. Two variables have a linear association if a scatter plot of the data can be well approximated by a line.

Linear equation. Any equation that can be written in the form $Ax + By + C = 0$ where A and B cannot both be 0. The graph of such an equation is a line.

Linear function. A function with an equation of the form $y = mx + b$, where m and b are constants

Logarithm. The exponent that indicates the power to which a base number is raised to produce a given number. *For example, the logarithm of 100 to the base 10 is 2.* (M)

Logarithmic function. Any function in which an independent variable appears in the form of a logarithm; they are the inverse functions of exponential functions.

M

Matrix (pl. matrices). A rectangular array of numbers or variables.

Mean. A measure of center in a set of numerical data, computed by adding the values in a list and then dividing by the number of values in the list.[33] *Example: For the data set {1, 3, 6, 7, 10, 12, 14, 15, 22, 120}, the mean is 21.*

Mean absolute deviation. A measure of variation in a set of numerical data, computed by adding the distances between each data value and the mean, then dividing by the number of data values. *Example: For the data set {2, 3, 6, 7, 10, 12, 14, 15, 22, 120}, the mean absolute deviation is 20.*

Measure of variability. A determination of how much the performance of a group deviates from the mean or median, most frequently used measure is standard deviation.

Median. A measure of center in a set of numerical data. The median of a list of values is the value appearing at the center of a sorted version of the list; or the mean of the two central values, if the list contains an even number of values. *Example: For the data set {2, 3, 6, 7, 10, 12, 14, 15, 22, 90}, the median is 11.*

Midline. In the graph of a trigonometric function, the horizontal line halfway between its maximum and minimum values.

Model. A mathematical representation (e.g., number, graph, matrix, equation(s), geometric figure) for real-world or mathematical objects, properties, actions, or relationships. (DPI)

Modulus of a complex number. The distance between a complex number and the origin on the complex plane. The absolute value of $a + bi$ is written $|a + bi|$, and the formula for $|a + bi|$ is $\sqrt{a^2+b^2}$. For a complex number in polar form, $r(\cos \theta + i \sin \theta)$, the modulus is r. (MW)

Multiplication and division within 100. Multiplication or division of two whole numbers with whole number answers, and with product or dividend in the range 0–100. *Example:* $72 \div 8 = 9$.

Multiplicative inverses. Two numbers whose product is 1 are multiplicative inverses of one another. *Example:* $\frac{3}{4}$ *and* $\frac{4}{3}$ *are multiplicative inverses of one another because* $\frac{3}{4} \times \frac{4}{3} = \frac{4}{3} \times \frac{3}{4} = 1$.

N

Network. a) A figure consisting of vertices and edges that shows how objects are connected; b) A collection of points (vertices), with certain connections (edges) between them.

Non-linear association. The relationship between two variables is nonlinear if the change in the second is not simply proportional to the change in the first, independent of the value of the first variable.

Number line diagram. A diagram of the number line used to represent numbers and support reasoning about them. In a number line diagram for measurement quantities, the interval from 0 to 1 on the diagram represents the unit of measure for the quantity.

Numeral. A symbol or mark used to represent a number.

O

Observational study. A type of study in which an action or behavior is observed in such a manner that no interference with, or influence upon, the behavior occurs.

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[33] To be more precise, this defines the *arithmetic mean*.

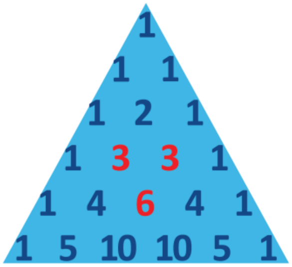
Order of Operations. Convention adopted to perform mathematical operations in a consistent order. 1. Perform all operations inside parentheses, brackets, and/or above and below a fraction bar in the order specified in steps 3 and 4; 2. Find the value of any powers or roots; 3. Multiply and divide from left to right; 4. Add and subtract from left to right. (NCTM)

Ordinal number. A number designating the place (as first, second, or third) occupied by an item in an ordered sequence. (M)

P

Partition. A process of dividing an object into parts.

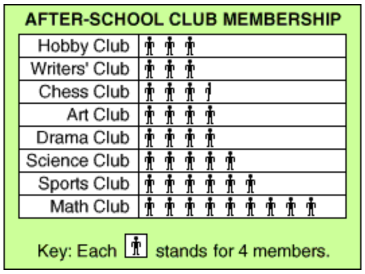
Pascal's triangle. A triangular arrangement of numbers in which each row starts and ends with 1, and each other number is the sum of the two numbers above it. (H)



Percent rate of change. A rate of change expressed as a percent. *Example: if a population grows from 50 to 55 in a year, it grows by 5/50 = 10% per year.*

Periodic phenomena. Naturally recurring events, for example, ocean tides, machine cycles.

Picture graph. A graph that uses pictures to show and compare information.



Plane. A flat surface that extends infinitely in a all directions.

Polar form. The polar coordinates of a complex number on the complex plane. The polar form of a complex number is written in any of the following forms: $r\cos\theta + r\,i\sin\theta$, $r(\cos\theta + i\sin\theta)$, or $r\operatorname{cis}\theta$. In any of these forms, r is called the modulus or absolute value. θ is called the argument. (MW)

Polynomial. The sum or difference of terms which have variables raised to positive integer powers and which have coefficients that may be real or complex. The following are all polynomials: $5x^3 – 2x^2 + x – 13$, $x^2y^3 + xy$, and $(1 + i)a^2 + ib^2$. (MW)

Polynomial function. Any function whose value is the solution of a polynomial.

Postulate. A statement accepted as true without proof.

Prime factorization. A number written as the product of all its prime factors. (H)

Prime number. A whole number greater than 1 whose only factors are 1 and itself.

Probability distribution. The set of possible values of a random variable with a probability assigned to each.

Properties of equality. See Table 4 in the Glossary.

Properties of inequality. See Table 6 in the Glossary.

Properties of operations. See Table 3 in the Glossary.

Probability. A number between 0 and 1 used to quantify likelihood for processes that have uncertain outcomes (such as tossing a coin, selecting a person at random from a group of people, tossing a ball at a target, testing for a medical condition).

Probability model. A probability model is used to assign probabilities to outcomes of a chance process by examining the nature of the process. The set of all outcomes is called the sample space, and their probabilities sum to 1. *See also:* ***uniform probability model***.

Proof. A proof of a mathematical statement is a detailed explanation of how that statement follows logically from statements already accepted as true.

Proportion. An equation that states that two ratios are equivalent, e.g., $4/8 = ½$ or $4 : 8 = 1 : 2$.

Pythagorean Theorem. For any right triangle, the sum of the squares of the measures of the legs equals the square of the measure of the hypotenuse.

Q

Quadratic equation. An equation that includes only second degree polynomials. Some examples are $y = 3x^2 – 5x^2 + 1$, $x^2 + 5xy + y^2 = 1$, and $1.6a^2 +5.9a – 3.14 = 0$. (MW)

Quadratic expression. An expression that contains the square of the variable, but no higher power of it.

Quadratic function. A function that can be represented by an equation of the form $y = ax^2 + bx + c$, where a, b, and c are arbitrary, but fixed, numbers and a 0. The graph of this function is a parabola. (DPI)

Quadratic polynomial. A polynomial where the highest degree of any of its terms is 2.

R

Radical. The $\sqrt{}$ symbol, which is used to indicate square roots or *n*th roots. (MW)

Random sampling. A smaller group of people or objects chosen from a larger group or population by a process giving equal chance of selection to all possible people or objects. (H)

Random variable. An assignment of a numerical value to each outcome in a sample space. (M)

Ratio. A relationship between quantities such that for every *a* units of one quantity there are *b* units of the other. A ratio is often denoted by *a:b* and read "*a to b*".

Rational expression. A quotient of two polynomials with a non-zero denominator.

Rational number. A number expressible in the form $^a/_b$ or $-^a/_b$ for some fraction $^a/_b$. The rational numbers include the integers. See Illustration 1 in the Glossary.

Real number. A number from the set of numbers consisting of all rational and all irrational numbers. See Illustration 1 in the Glossary.

Rectangular array. An arrangement of mathematical elements into rows and columns.

Rectilinear figure. A polygon all angles of which are right angles.

Recursive pattern or sequence. A pattern or sequence wherein each successive term can be computed from some or all of the preceding terms by an algorithmic procedure.

Reflection. A type of transformation that flips points about a line, called the *line of reflection.* Taken together, the image and the pre-image have the line of reflection as a line of symmetry.

Relative frequency. The empirical counterpart of probability. If an event occurs N' times in N trials, its relative frequency is $^{N'}/_{N}$. (M)

Relatively Prime. Two positive integers that share no common divisors greater than 1; that is, the only common positive factor of the two numbers is 1.

Remainder Theorem. If $f(x)$ is a polynomial in x then the remainder on dividing $f(x)$ by $x - a$ is $f(a)$. (M)

Repeating decimal. A decimal in which, after a certain point, a particular digit or sequence of digits repeats itself indefinitely; the decimal form of a rational number. (M) *See also:* ***terminating decimal****.*

Rigid motion. A transformation of points in space consisting of a sequence of one or more translations, reflections, and/or rotations. Rigid motions are here assumed to preserve distances and angle measures.

Rotation. A type of transformation that turns a figure about a fixed point, called the *center of rotation*.

S

SAS congruence (Side-angle-side congruence). When two triangles have corresponding sides and the angles formed by those sides are congruent, the triangles are congruent. (MW)

SSS congruence (Side-side-side congruence). When two triangles have corresponding sides that are congruent, the triangles are congruent. (MW)

Sample space. In a probability model for a random process, a list of the individual outcomes that are to be considered.

Scatter plot. A graph in the coordinate plane representing a set of bivariate data. *For example, the heights and weights of a group of people could be displayed on a scatter plot.* (DPI)

Scientific notation. A widely used floating-point system in which numbers are expressed as products consisting of a number between 1 and 10 multiplied by an appropriate power of 10, e.g., $562 = 5.62 \times 10^2$. (MW)

Sequence, progression. A set of elements ordered so that they can be labeled with consecutive positive integers starting with 1, e.g., 1, 3, 9, 27, 81. In this sequence, 1 is the *first term*, 3 is the *second term*, 9 is the *third term*, and so on.

Significant figures (digits). A way of describing how precisely a number is written, particularly when the number is a measurement. (MW)

Similarity transformation. A rigid motion followed by a dilation.

Simultaneous equations. Two or more equations containing common variables. (MW)

Sine (of an acute angle). The trigonometric function that for an acute angle is the ratio between the leg opposite the angle when the angle is considered part of a right triangle and the hypotenuse. (M)

T

Tangent. a) Meeting a curve or surface in a single point if a sufficiently small interval is considered. b) The trigonometric function that, for an acute angle, is the ratio between the leg opposite the angle and the leg adjacent to the angle when the angle is considered part of a right triangle. (MW)

Tape diagram. A drawing that looks like a segment of tape, used to illustrate number relationships. Also known as a strip diagram, bar model, fraction strip, or length model.

Terminating decimal. A decimal is called terminating if its repeating digit is 0. A terminating decimal is the decimal form of a rational number. *See also: **repeating decimal***.

Third quartile. For a data set with median *M*, the third quartile is the median of the data values greater than *M*. *Example: For the data set {2, 3, 6, 7, 10, 12, 14, 15, 22, 120}, the third quartile is 15. See also: **median, first quartile, interquartile range**.*

Transformation. A prescription, or rule, that sets up a one-to-one correspondence between the points in a geometric object (the *pre-image*) and the points in another geometric object (the *image*). Reflections, rotations, translations, and dilations are particular examples of transformations.

Transitivity principle for indirect measurement. If the length of object A is greater than the length of object B, and the length of object B is greater than the length of object C, then the length of object A is greater than the length of object C. This principle applies to measurement of other quantities as well.

Translation. A type of transformation that moves every point in a graph or geometric figure by the same distance in the same direction without a change in orientation or size. (MW)

Trapezoid. A quadrilateral with at least one pair of parallel sides. (Note: There are two definitions for the term trapezoid. This is the inclusive definition. For more information see *commoncoretools.me/wpcontent/uploads/2014/12/ccss_progression_gk6_2014_12_27.pdf*).

Trigonometric function. A function (as the sine, cosine, tangent, cotangent, secant, or cosecant) of an arc or angle most simply expressed in terms of the ratios of pairs of sides of a right-angled triangle. (M)

Trigonometry. The study of triangles, with emphasis on calculations involving the lengths of sides and the measure of angles. (MW)

U

Uniform probability model. A probability model which assigns equal probability to all outcomes. *See also: **probability model**.*

Unit fraction. A fraction with a numerator of 1, such as 1/3 or 1/5.

V

Valid. a) Well-grounded or justifiable; being at once relevant and meaningful, e.g., a *valid* theory; b) Logically correct. (MW)

Variable. A quantity that can change or that may take on different values. Refers to the letter or symbol representing such a quantity in an expression, equation, inequality, or matrix. (MW)

Vector. A quantity with magnitude and direction in the plane or in space, defined by an ordered pair or triple of real numbers.

Visual fraction model. A tape diagram, number line diagram, or area model.

W

Whole numbers. The numbers 0, 1, 2, 3,…. See Illustration 1 in the Glossary.

Tables and Illustrations of Key Mathematical Properties, Rules, and Number Sets

Table 1. Common addition and subtraction situations[34]

	Result Unknown	**Change Unknown**	**Start Unknown**
Add to	Two bunnies sat on the grass. Three more bunnies hopped there. How many bunnies are on the grass now? 2 + 3 = ?	Two bunnies were sitting on the grass. Some more bunnies hopped there. Then there were five bunnies. How many bunnies hopped over to the first two? 2 + ? = 5	Some bunnies were sitting on the grass. Three more bunnies hopped there. Then there were five bunnies. How many bunnies were on the grass before? ? + 3 = 5
Take from	Five apples were on the table. I ate two apples. How many apples are on the table now? 5 – 2 = ?	Five apples were on the table. I ate some apples. Then there were three apples. How many apples did I eat? 5 – ? = 3	Some apples were on the table. I ate two apples. Then there were three apples. How many apples were on the table before? ? – 2 = 3

	Total Unknown	**Addend Unknown**	**Both Addends Unknown[35]**
Put Together/ Take Apart[36]	Three red apples and two green apples are on the table. How many apples are on the table? 3 + 2 = ?	Five apples are on the table. Three are red and the rest are green. How many apples are green? 3 + ? = 5, 5 – 3 = ?	Grandma has five flowers. How many can she put in her red vase and how many in her blue vase? 5 = 0 + 5, 5 = 5 + 0 5 = 1 + 4, 5 = 4 + 1 5 = 2 + 3, 5 = 3 + 2

	Difference Unknown	**Bigger Unknown**	**Smaller Unknown**
Compare[37]	(“How many more?” version): Lucy has two apples. Julie has five apples. How many more apples does Julie have than Lucy? (“How many fewer?” version): Lucy has two apples. Julie has five apples. How many fewer apples does Lucy have than Julie? 2 + ? = 5, 5 – 2 = ?	(Version with “more”): Julie has three more apples than Lucy. Lucy has two apples. How many apples does Julie have? (Version with “fewer”): Lucy has 3 fewer apples than Julie. Lucy has two apples. How many apples does Julie have? 2 + 3 = ?, 3 + 2 = ?	(Version with “more”): Julie has three more apples than Lucy. Julie has five apples. How many apples does Lucy have? (Version with “fewer”): Lucy has three fewer apples than Julie. Julie has five apples. How many apples does Lucy have? 5 – 3 = ?, ? + 3 = 5

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[34] Adapted from Boxes 2–4 of *Mathematics Learning in Early Childhood*, National Research Council (2009, pp. 32–33).

[35] These take apart situations can be used to show all the decompositions of a given number. The associated equations, which have the total on the left of the equal sign, help children understand that the = sign does not always mean *makes* or *results in* but always does mean *is the same number as*.

[36] Either addend can be unknown, so there are three variations of these problem situations. Both Addends Unknown is a productive extension of this basic situation, especially for small numbers less than or equal to 10.

[37] For the Bigger Unknown or Smaller Unknown situations, one version directs the correct operation (the version using *more* for the bigger unknown and using *less* for the smaller unknown). The other versions are more difficult.

Table 2. Common multiplication and division situations[38]

	Unknown Product **3 × 6 = ?**	**Group Size Unknown** (“How many in each group?” Division) **3 × ? = 18 and 18 ÷ 3 = ?**	**Number of Groups Unknown** (“How many groups?” Division) **? × 6 = 18 and 18 ÷ 6 = ?**
Equal Groups	There are three bags with six plums in each bag. How many plums are there in all? *Measurement example*. You need three lengths of string, each six inches long. How much string will you need altogether?	If 18 plums are shared equally into three bags, then how many plums will be in each bag? *Measurement example*. You have 18 inches of string, which you will cut into three equal pieces. How long will each piece of string be?	If eighteen plums are to be packed six to a bag, then how many bags are needed? *Measurement example*. You have 18 inches of string, which you will cut into pieces that are six inches long. How many pieces of string will you have?
Arrays,[39] Area[40]	There are three rows of apples with six apples in each row. How many apples are there? *Area example*. What is the area of a 3 cm by 6 cm rectangle?	If 18 apples are arranged into three equal rows, how many apples will be in each row? *Area example*. A rectangle has area 18 square centimeters. If one side is 3 cm long, how long is a side next to it?	If 18 apples are arranged into equal rows of six apples, how many rows will there be? *Area example*. A rectangle has area 18 square centimeters. If one side is 6 cm long, how long is a side next to it?
Compare	A blue hat costs $6. A red hat costs three times as much as the blue hat. How much does the red hat cost? *Measurement example*. A rubber band is 6 cm long. How long will the rubber band be when it is stretched to be three times as long?	A red hat costs $18 and that is three times as much as a blue hat costs. How much does a blue hat cost? *Measurement example*. A rubber band is stretched to be 18 cm long and that is three times as long as it was at first. How long was the rubber band at first?	A red hat costs $18 and a blue hat costs $6. How many times as much does the red hat cost as the blue hat? *Measurement example*. A rubber band was 6 cm long at first. Now it is stretched to be 18 cm long. How many times as long is the rubber band now as it was at first?
General	$a \times b = ?$	$a \times ? = p$ and $p \div a = ?$	$? \times b = p$ and $p \div b = ?$

---

[38] The first examples in each cell are examples of discrete things. These are easier for students and should be given before the measurement examples.

[39] The language in the array examples shows the easiest form of array problems. A harder form is to use the terms rows and columns: The apples in the grocery window are in three rows and six columns. How many apples are in there? Both forms are valuable.

[40] Area involves arrays of squares that have been pushed together so that there are no gaps or overlaps, so array problems include these especially important measurement situations.

Table 3. The Properties of Operations

Here a, b and c stand for arbitrary numbers in a given number system. The properties of operations apply to the rational number system, the real number system, and the complex number system.

Associative property of addition	$(a + b) + c = a + (b + c)$
Commutative property of addition	$a + b = b + a$
Additive identity property of 0	$a + 0 = 0 + a = a$
Existence of additive inverses	For every a there exists $–a$ so that $a + (–a) = (–a) + a = 0$.
Associative property of multiplication	$(a \times b) \times c = a \times (b \times c)$
Commutative property of multiplication	$a \times b = b \times a$
Multiplicative identity property of 1	$a \times 1 = 1 \times a = a$
Existence of multiplicative inverses	For every $a \neq 0$ there exists $1/a$ so that $a \times 1/a = 1/a \times a = 1$.
Distributive property of multiplication over addition	$a \times (b + c) = a \times b + a \times c$

Table 4. The Properties of Equality

Here a, b, and c stand for arbitrary numbers in the rational, real, or complex number systems.

Reflexive property of equality	$a = a$
Symmetric property of equality	If $a = b$, then $b = a$.
Transitive property of equality	If $a = b$ and $b = c$, then $a = c$.
Addition property of equality	If $a = b$, then $a + c = b + c$.
Subtraction property of equality	If $a = b$, then $a – c = b – c$.
Multiplication property of equality	If $a = b$, then $a \times c = b \times c$.
Division property of equality	If $a = b$ and $c \neq 0$, then $a \div c = b \div c$.
Substitution property of equality	If $a = b$, then b may be substituted for a in any expression containing a.

Table 5. Algorithms and the Standard Algorithms: Addition Example

Algorithm	**Standard Algorithm (for efficiency)**
356 +167 400 (Sum of hundreds) 110 (Sum of tens) 13 (Sum of ones) 523	11 356 +167 523
Note: *All algorithms have a finite set of steps, are based on place value and properties of operations, and use single-digit computations.*	

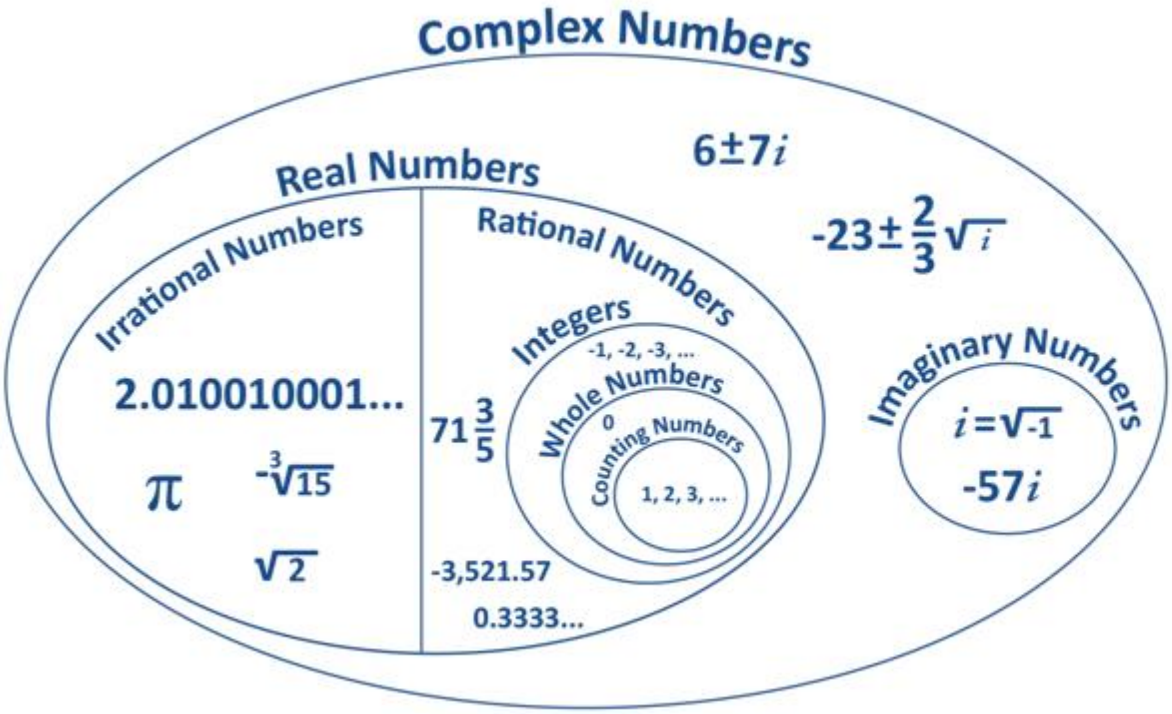
Table 6. The Properties of Inequality

Here a, b, and c stand for arbitrary numbers in the rational or real number systems.

Exactly one of the following is true: $a < b$, $a = b$, $a > b$.
If $a > b$ and $b > c$ then $a > c$.
If $a > b$, then $b < a$.
If $a > b$, then $-a < -b$.
If $a > b$, then $a \pm c > b \pm c$.
If $a > b$ and $c > 0$, then $a \times c > b \times c$.
If $a > b$ and $c < 0$, then $a \times c < b \times c$.
If $a > b$ and $c > 0$, then $a \div c > b \div c$.
If $a > b$ and $c < 0$, then $a \div c < b \div c$.

Illustration 1. The Number System

The Number System is comprised of number sets beginning with the Counting Numbers and culminating in the more complete Complex Numbers. The name of each set is written on the boundary of the set, indicating that each increasing oval encompasses the sets contained within. Note that the Real Number Set is comprised of two parts: Rational Numbers and Irrational Numbers.



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